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ORIGINAL ARTICLE

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Motion equation for a flexible one-dimensional element used in the dynamical analysis of a multibody system

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Abstract In this study, the motion equations of a one-dimensional finite element having a general three-dimensional motion together the body are established, using the Lagrange's equations. The problem is important in technical applications of the last decades, characterized by high velocities and high applied loads. This leads to qualitative different mechanical phenomena (high deformations, resonance, stability), mainly due to the Coriolis effects and relative motions.

Keywords Multi-body system · Finite element method · Linear elastic elements · Lagrange's equations · Three-dimensional motion · One-dimensional finite element

1 Introduction

In the case when the velocities or the load involved in a multibody system (MBS) become high, the hypothesis of rigid elements may not correspond to the reality and the mathematical models must be improved considering the elasticity of the elements. For example, the phenomena of resonance and loss of stability represent classic forms of manifestation of the elasticity properties. These phenomena can influence, generally unfavorable, the approach of such a system. The classic form to approach this type of problems is to apply continuous mathematical models, but this type of analysis is not very useful for practical purposes. The major disadvantage is the differential equations obtained, which cannot be solved easily even if numerical methods are used (see [10–13]). A more convenient way is represented by numerical method. Among these methods, the most powerful approach is the finite element method (FEM). The advantages of this approach result from [1,2,5,9,19,22,23,25–28].

The papers approaching this field have performed an analysis of a single deformable element, having a plane motion, and then the study was extended to the mechanisms with plane-parallel motion [3,14,17,20,28] with deformable elements. In [4,7,8,21], the results obtained in this field are being synthesized. More complex models studied some particular effects [6,18].

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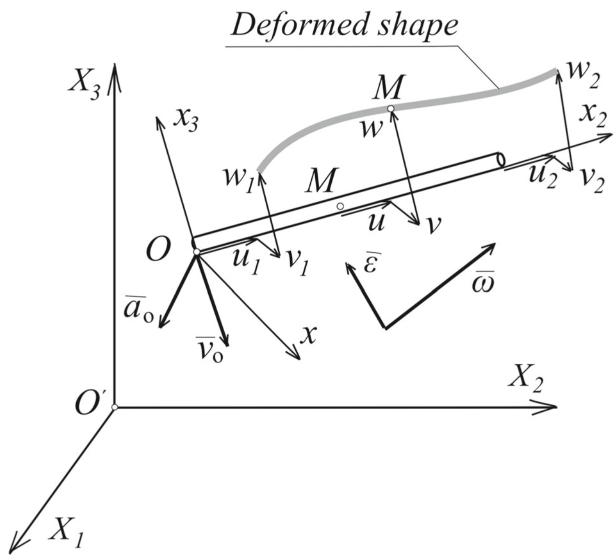

Fig. 1 One-dimensional finite element

The objective of this paper is to provide a theoretical background for further development of FEM applied to the study of MBS. A model for a one-dimensional finite element with a general three-dimensional motion is developed. Furthermore, the shear effect on bending is taken into account.

2 Lagrangian of an element

Let us consider in the following a truss finite element participating with the body to the general rigid body motion of a multibody system (Fig. 1). To obtain the governing motion equation for a single element, we use the method of Lagrange's equations. The first step in such an approach is to obtain the Lagrangian for a truss element (which is able to have bending, traction–compression and torsion). For this, it shall be computed the kinetic energy of the considered finite element, the internal energy and the work of the distributed and concentrated loads (the constitutive parts of the Lagrangian).

For one single finite element, the generalized independent coordinates will be the nodal displacements of the two ends (which may include the displacements of the bar ends in the three directions and the derivatives of these values having the geometrical significance of rotations and curvatures). The number of independent coordinates depends on the hypothesis used and will determine the expression of the shape functions.

Within the part of the multibody system containing the considered finite element, the latter has a general rigid body motion. It is assumed that the rigid motion of the mechanical system was previously determined using a dynamical analysis considering all the elements of the multibody system being rigid. We make the assumption that the elastic deformations of the elements are small and would not influence the general motion of the multibody system. Thus, it is assumed that the field of velocities and accelerations for each constitutive part of the mechanical system is known.

Consider now a truss finite element, having at its ends the nodes numbered i and j. The finite element refers to a local reference system $Oxyz$, participating to the general three-dimensional rigid motion of the truss (Fig. 1). We consider that the velocity and the acceleration of the origin of the local coordinate reference are known. The angular velocity and the angular acceleration of the local coordinate system are also considered as being known. Index G applied to a physical size (a scalar, vector or matrix) indicates that the value is expressed in a global reference system. The index L indicates that the value is related to the local reference system.

The displacement $\delta(u, v, w)$ of an arbitrary point M chosen at a distance x from the left-hand end of the bar can be written, using the shape functions N_{ij} and the vector of the nodal displacements, in the local coordinate system:

$$u = \delta_1 = N_{1j}\delta_{e,j}, \quad v = \delta_2 = N_{2j}\delta_{e,j}, \quad w = \delta_3 = N_{3j}\delta_{e,j}, \quad j = \overline{1, 12}, \tag{1}$$

or, in a compact form:

$$\delta_i = N_{ij}\delta_{e,j}, \quad i = 1, 2, 3, \quad j = \overline{1, 12}. \tag{2}$$

The nodal displacement vector of the finite element numbered e, δ_e, is:

$$\delta_e^T = \left[\delta_1^{(1)}\ \delta_2^{(1)}\ \delta_3^{(1)}\ L\alpha^{(1)}\ L\beta^{(1)}\ L\gamma^{(1)}\ \vdots\ \delta_1^{(2)}\ \delta_2^{(2)}\ \delta_3^{(2)}\ L\alpha^{(2)}\ L\beta^{(2)}\ L\gamma^{(2)}\right]. \tag{3}$$

Here, we used the notations:

- $\delta_1^{(1)} = u_1$, $\delta_2^{(1)} = v_1$, $\delta_3^{(1)} = w_1$, i.e., the displacement of the left-hand end of the bar along the three directions;
- $\delta_1^{(2)} = u_2$, $\delta_2^{(2)} = v_2$, $\delta_3^{(2)} = w_2$, i.e., the displacement of the right-hand end of the bar along the three directions;
- $\alpha^{(1)} = \alpha_1,\ \beta^{(1)} = \beta_1,\ \gamma^{(1)} = \gamma_1$ are the rotations of the left-hand end section around the three axes;
- $\alpha^{(2)} = \alpha_2,\ \beta^{(2)} = \beta_2,\ \gamma^{(2)} = \gamma_2$ are the rotations of the right-hand end section around the three axes.

In the following, the three rows of the shape functions matrix N correspond to the displacements u, v and w and are named $N_{(u)} = N_{(1)}$, $N_{(v)} = N_{(2)}$ and $N_{(w)} = N_{(3)}$, so that we have

$$N = \begin{bmatrix} N_{(u)} \\ N_{(v)} \\ N_{(w)} \end{bmatrix} = \begin{bmatrix} N_{(1)} \\ N_{(2)} \\ N_{(3)} \end{bmatrix} = \left[N_{ij}\right], \quad i = 1, 2, 3, \quad j = \overline{1, 12}. \tag{4}$$

Usually, for α is adopted the same shape function as for the axial deformation:

$$\alpha = \delta_4 = N_{4i}\delta_{e,i}, \quad i = \overline{1, 12}. \tag{5}$$

We shall have the well-known equations from continuous mechanics Vlase [25]:

$$\beta = \delta_5 = -\frac{\mathrm{d}v}{\mathrm{d}x}, \qquad \gamma = \delta_6 = \frac{\mathrm{d}w}{\mathrm{d}x}. \tag{6}$$

The rotation of the bar section at the distance x from the left-hand end can be expressed as follows:

$$\beta = \frac{\mathrm{d}}{\mathrm{d}x}\left(N_{3i}\delta_{e,i}\right) = -N'_{3i}\delta_{e,i} = N_{5i}\delta_{e,i}, \qquad \gamma = \frac{\mathrm{d}}{\mathrm{d}x}\left(N_{2i}\delta_{e,i}\right) = N'_{2i}\delta_{e,i} = N_{6i}\delta_{e,i}, \tag{7}$$

$$\delta_i = N_{ij}\delta_{e,j}, \quad i = 4, 5, 6. \tag{8}$$

After deformation, the point $M(x_1, x_2, x_3)$ is moved to the point $M'(x'_1, x'_2, x'_3)$, where we used the notations:

$$x'_1 = x_1 + u = x_1 + \delta_1, \quad x'_2 = v = \delta_2, \quad x'_3 = w = \delta_3, \tag{9}$$

or, with respect to the global reference system:

$$\begin{aligned} X'_1 &= X_1 + r_{1i}\delta_i = X_{10} + r_{11}x_1 + r_{1i}\delta_i = X_{10} + r_{11}x_1 + r_{1i}N_{ij}\delta_{e,j}, \\ X'_2 &= X_2 + r_{2i}\delta_i = X_{20} + r_{21}x_1 + r_{2i}\delta_i = X_{20} + r_{21}x_1 + r_{2i}N_{ij}\delta_{e,j}, \\ X'_3 &= X_3 + r_{3i}\delta_i = X_{30} + r_{31}x_1 + r_{3i}\delta_i = X_{30} + r_{31}x_1 + r_{3i}N_{ij}\delta_{e,j}, \end{aligned} \tag{10}$$

where $i = 1, 2, 3$ and $j = \overline{1, 12}$.
Obviously, the system (10) can be rewritten in the following compact form:

$$X'_k = \alpha_{k1}x_1 + \alpha_{ki}N_{ij}\delta_{e1,j}, \quad k = \overline{1, 3}. \tag{11}$$

The velocity can be obtained after a differentiation with respect to the time variable:

$$\dot{X}'_k = \dot{X}_{k0} + \dot{r}_{k1}x_1 + \dot{r}_{ki}N_{ij}\delta_{e1,j} + r_{ki}N_{ij}\dot{\delta}_{e1,j}, \quad k = \overline{1, 3}. \tag{12}$$

For the entire bar, the kinetic energy due to the translation is:

$$\begin{aligned} E_{kt} &= \frac{1}{2}\int_0^L \varrho A \dot{X}'_k \dot{X}'_k \mathrm{d}x_1 \\ &= \frac{1}{2}\int_0^L \varrho A\big(\dot{X}_{k0} + \dot{r}_{k1}x_1 + \dot{r}_{ki}N_{ij}\delta_{e1,j} + r_{ki}N_{ij}\dot{\delta}_{e1,j}\big)\left(\dot{X}_{k0} + \dot{r}_{k1}x_1 + \dot{r}_{ki}N_{ij}\delta_{e1,j} + r_{ki}N_{ij}\dot{\delta}_{e1,j}\right)\,\mathrm{d}x_1. \end{aligned} \tag{13}$$

The kinetic energy of the infinitesimal element dm, due to the rotation is:

$$E_{kr} = \frac{1}{2}\int_0^L \varrho\ I_{ij}\ \omega'_i\ \omega'_j\ \mathrm{d}x, \tag{14}$$

where:

– (i) The infinitesimal element dm has the angular velocity $\{\omega'_L\}$ with the components:

$$\omega'_1 = \omega_1 + \dot{\alpha},\quad \omega'_2 = \omega_2 + \dot{\beta},\quad \omega'_3 = \omega_3 + \dot{\gamma}, \tag{15}$$

or, taking into account Eqs. (5) and (7)

$$\omega'_i = \omega_i + N_{i+3,j}\ \dot{\delta}_{e,j}. \tag{16}$$

From Eq. (8), we also deduce:

$$\dot{\alpha} = N_{4i}\ \dot{\delta}_{e,i},\quad \dot{\beta} = N_{5i}\ \dot{\delta}_{e,i},\quad \dot{\gamma} = N_{6i}\ \dot{\delta}_{e,i}. \tag{17}$$

In (15) and (16), we denoted by $\omega_1, \omega_2, \omega_3$ the components of the angular velocity vector related to the local (mobile) reference system.

– (ii) The matrix $I = I_{ij}$ has the components:

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & -I_{yz} \\ 0 & -I_{yz} & I_{zz} \end{bmatrix}, \tag{18}$$

where I_{yy} and I_{zz} represent the moments of inertia of the bar cross section about the axes Oy and Oz, respectively. The reference system has its origin in the mass center of the element $dm = \varrho A\mathrm{d}x$ (ϱ - density), I_{yz} is the centrifugal moment of inertia, and I_{xx} is the inertia moment about the axis Ox. Since we have chosen y and z as principal directions of inertia $I_{yz} = 0$, the matrix of moments of inertia becomes:

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}, \tag{19}$$

where, for the sake of simplicity, we used the convention: $I_{xx} = I_x$, $I_{yy} = I_y$, $I_{zz} = I_z$.
The coefficient r_{ij} defines the position of the unit vector of the local coordinate system $Oxyz$. The derivatives represent the angular velocities and accelerations. For a better understanding, we shall express these derivatives according to rule of Simeon in [22]. The orthogonality condition of the unit vectors leads to:

$$r_{ik}r_{kj} = r_{jk}r_{ki} = \delta_{ij}, \tag{20}$$

where δ_{ij} is the Kronecker's symbol. If we differentiate this equation, one obtains:

$$\dot{r}_{ik}r_{kj} + r_{ik}\dot{r}_{kj} = 0,\quad i, j = 1, 2, 3. \tag{21}$$

If we use the notation $\Omega_{ij} = \dot{r}_{ik}r_{kj}$, then the relation (21) receives the form:

$$\Omega_{ij} + \Omega_{ji} = 0. \tag{22}$$

Here, Ω_{ij} is the operator of the angular velocity and, according to Eq. (22), Ω_{ij} is a skew-symmetric tensor. We must outline that the components of the tensor Ω_{ij} are expressed in the global reference system. To this tensor, we can associate an angular velocity vector whose components are defined by means of the following equations:

$$\Omega_1 = \Omega_{32} = -\Omega_{23}, \quad \Omega_2 = \Omega_{13} = -\Omega_{31}, \quad \Omega_3 = \Omega_{21} = -\Omega_{12}. \tag{23}$$

We shall also have the angular acceleration skew-symmetric operator, having the components:

$$E_{ij} = \dot{\Omega}_{ij} = \ddot{r}_{ik} r_{kj} + \dot{r}_{ik} \dot{r}_{kj}. \tag{24}$$

To this operator, we associate the angular acceleration vector, having the components:

$$E_1 = E_{32} = -E_{23}, \quad E_2 = E_{13} = -E_{31}, \quad E_3 = E_{21} = -E_{12}. \tag{25}$$

Thus, after performing some elementary calculations, we shall obtain:

$$E_{ij} = \dot{\Omega}_{ij} = \ddot{r}_{ik} r_{kj} + \dot{r}_{ik} \dot{r}_{kj} = \ddot{r}_{ik} r_{kj} + \dot{r}_{ik} r_{kl} r_{lm} \dot{r}_{mj} = \ddot{r}_{ik} r_{kj} - \Omega_{ik} \Omega_{kj}, \tag{26}$$

from where we deduce that

$$\ddot{r}_{ik} r_{kj} = E_{ij} + \Omega_{ik} \Omega_{kj}. \tag{27}$$

This last result will be used in the following calculus.
The components of the angular velocity and of the acceleration vectors can be expressed in the local reference system by using the relations:

$$\omega_i = r_{ij} \omega_j, \quad \varepsilon_i = r_{ij} \varepsilon_j, \quad i = 1, 2, 3. \tag{28}$$

Also, for the components of the angular velocity and of the acceleration we will use the expressions:

$$\omega_{ij} = r_{ki} \omega_{km} r_{mj}, \quad \varepsilon_{ij} = r_{ki} \varepsilon_{km} r_{mj}, \quad i = 1, 2, 3. \tag{29}$$

Next shall be calculated the internal energy stored in the bar. The energy of the bar due to bending is:

$$\begin{aligned} E_{pb} &= \frac{1}{2} \int_0^L \left[EI_y \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right)^2 + EI_z \left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \right)^2 \right] \mathrm{d}x \\ &= \frac{1}{2} \int_0^L \left[EI_y \beta'^2 + EI_z \gamma'^2 \right] \mathrm{d}x. \end{aligned} \tag{30}$$

We introduce the expression of w and v of the form:

$$w = \delta_3 = N_{3i} \delta_{e,i}, \quad v = \delta_{L,2} = N_{2i} \delta_{eL,i}, \tag{31}$$

in the relation of the internal energy (30) so that it receives the form:

$$E_{pb} = \frac{1}{2} \delta_{e,i} \left[\int_0^L \left[EI_y N''_{3i} N''_{3j} + EI_z N''_{2i} N''_{2j} \right] \mathrm{d}x \right] \delta_{e,j} = \frac{1}{2} \delta_{e,i} k_{b,ij} \delta_{e,j}, \tag{32}$$

where

$$k_{b,ij} = \int_0^L \left[EI_y N''_{3i} N''_{3j} + EI_z N''_{2i} N''_{2j} \right] \mathrm{d}x. \tag{33}$$

Consider now the energy caused by the axial deformation:

$$E_{pa} = \frac{1}{2} \int_0^L EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x = \frac{1}{2} \delta_{eL,i} \left(\int_0^L EA N'_{1i} N'_{1j} \, \mathrm{d}x \right) \delta_{eL,j} = \frac{1}{2} \delta_{e,i} k_{a,ij} \delta_{e,j}, \tag{34}$$

where

$$k_{a,ij} = \int_0^L EA N'_{1i} N'_{1j} \, \mathrm{d}x. \tag{35}$$

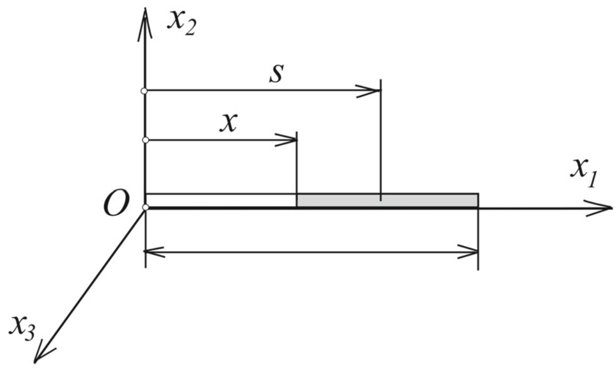

Fig. 2 Determination of the axial inertia force

In a similar way, we obtain the deformation energy due to torsion:

$$E_{pt} = \frac{1}{2}\int_0^L GI_x \left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{1}{2}\delta_{eL,i}\left(\int_0^L GI_x N'_{4i}N'_{4j}\,\mathrm{d}x\right)\delta_{eL,j} = \frac{1}{2}\delta_{e,i}k_{t,ij}\delta_{e,j}, \tag{36}$$

where

$$k_{t,ij} = \int_0^L GI_x N'_{4i}N'_{4j}\,\mathrm{d}x. \tag{37}$$

An axial load P_{tot} existing in an axial section of the bar gives the following energy if, in a first approximation, the axial deformations are neglected:

$$E_a = \frac{1}{2}\int_0^L P_{\mathrm{tot}}\left[\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 + \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2\right]\mathrm{d}x, \tag{38}$$

where P_{tot} represents the axial force in the bar cross section at the distance x. We are interested to determine the effect of the inertia forces acting along the bar. We consider that the force components acting at the right-hand bar end are, in the local coordinate system, P_x, $P_y = 0$, $P_z = 0$. Beside these components, we will determine the components of the inertia forces acting upon the portion of the bar between x and L. The acceleration of a current point of the bar with the abscissa x is (Fig. 2):

$$a_{G,1} = a_{01,G} + (r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\,x - \left[r_{11}\left(\omega_2^2 + \omega_3^2\right) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]x. \tag{39}$$

The inertia force is given by:

$$\begin{aligned} F_{i,1} &= -\int_x^L a_{G,1}\mathrm{d}m = -\int_x^L a_{0x,G}\varrho A\mathrm{d}s - \int_x^L \varrho A\,(r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\,s\mathrm{d}s \\ &\quad + \int_x^L \varrho A\left[r_{11}\left(\omega_2^2+\omega_3^2\right) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]s\mathrm{d}s \\ &= -\ddot{X}_0\varrho A(L-x) - \frac{1}{2}\varrho A\,(r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\left(L^2 - x^2\right) \\ &\quad + \frac{1}{2}\varrho A\left[r_{11}\left(\omega_2^2+\omega_3^2\right) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]\left(L^2 - x^2\right). \end{aligned} \tag{40}$$

By using the notations:

$$\begin{aligned} \mu &= -\ddot{X}_0\varrho AL - \frac{1}{2}\varrho AL^2(r_{12}\varepsilon_3 - r_{13}\varepsilon_2) + \frac{1}{2}\varrho AL^2\left[r_{11}\left(\omega_2^2+\omega_3^2\right) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right], \\ \lambda &= \varrho A\ddot{X}_0, \quad \nu = \frac{1}{2}\varrho A\,(r_{12}\varepsilon_3 - r_{13}\varepsilon_2) - \frac{1}{2}\varrho A\left[r_{11}\left(\omega_2^2+\omega_3^2\right) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right], \end{aligned} \tag{41}$$

the internal energy due to the inertia of the bar mass receives the form:

$$\begin{aligned} E_a &= \frac{1}{2}\delta_{eL,i}\left[\int_0^L \left(P_x+\mu_x+\lambda_x x+\nu_x x^2\right)\left(N_{3i}^* N_{3j}^* + N_{2i}^* N_{2j}^*\right)\mathrm{d}x_1\right]\delta_{eL,j} \\ &= \frac{1}{2}\delta_{eL,i}\left[\int_0^L \left(P_x+\mu_x+\lambda_x x+\nu_x x^2\right)\left(N_{2i} N_{2j} + N_{3i} N_{3j}^*\right)\mathrm{d}x_1\right]\delta_{eL,j} \\ &= \frac{1}{2}\delta_{e,i}k_{ij}^G\delta_{e,j}, \end{aligned} \tag{42}$$

where

$$k_{ij}^G = \int_0^L \left(P_x+\mu_x+\lambda_x x+\nu_x x^2\right)\left(N_{2i} N_{2j} + N_{3i} N_{3j}^*\right)\mathrm{d}x_1. \tag{43}$$

By using the notation

$$k_{e,ij} = k_{b,ij} + k_{a,ij} + k_{t,ij} + k_{ij}^G, \tag{44}$$

we can write the total internal energy in the form:

$$E_p = \frac{1}{2}\delta_{e,i}\left(k_{b,ij}+k_{a,ij}+k_{t,ij}+k_{ij}^G\right)\delta_{e,j} = \frac{1}{2}\delta_{e,i}k_{e,ij}\delta_{e,j}. \tag{45}$$

The external mechanical work of the concentrated loads with the local components $q_{e,i}$, applied in the nodes, is:

$$W^c = q_{e,i}\delta_{e,i}. \tag{46}$$

Also, the external mechanical work of the distributed loads is:

$$\begin{aligned} W &= \int_0^L \left(p_1\delta_{L,1}+p_2\delta_{L,2}+p_3\delta_{L,3}+m_1\alpha+m_2\beta+m_3\gamma\right)\mathrm{d}x \\ &= \left(\int_0^L p_i N_{ij}\mathrm{d}x\right)\delta_{e,j} + \left(\int_0^L m_i N_{(i+3)j}\mathrm{d}x\right)\delta_{e,j} = q_{e,j}^*\delta_{e,i}, \quad i=1,2,3, \quad j=\overline{1,12}, \end{aligned} \tag{47}$$

where we used the notation

$$q_{e,j}^* = \left(\int_0^L p_i N_{ij}\mathrm{d}x\right)\delta_{e,j} + \left(\int_0^L m_i N_{(i+3)j}\mathrm{d}x\right)\delta_{e,j}, \quad i=1,2,3, \quad j=\overline{1,12}. \tag{48}$$

The Lagrangian of the element becomes:

$$L = E_c - E_p + W^d + W^c. \tag{49}$$

Thus, by taking into account Eqs. (32), (34), (36), (42), (44)–(48), we obtain:

$$\begin{aligned} L = {} & \frac{1}{2}\int_0^L \varrho A\left(\dot{X}_{k0}+\dot{r}_{k1}x_1+\dot{r}_{ki}N_{ij}\delta_{eL,j}+r_{ki}N_{ij}\dot{\delta}_{eL,j}\right) \\ & \times\left(\dot{X}_{k0}+\dot{r}_{k1}x_1+\dot{r}_{ki}N_{ij}\delta_{eL,j}+r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\mathrm{d}x_1 \\ & -\frac{1}{2}\delta_{eL,i}k_{e,ij}\delta_{eL,j} + q_{eL,j}^*\delta_{eL,j} + q_{eL,i}\delta_{eL,i}. \end{aligned} \tag{50}$$

For details, see [16,27,30,31].

3 Motion equations

We begin this section with a result regarding the equations of motion.

Theorem 1 *The motion equations written in the local coordinate system for one-dimensional finite element take the form*

$$m_{e,ij}\ddot{\delta}_{eL,j}+2c^{\omega}_{e,ij}\dot{\delta}_{eL,j}+\left(k_{e,ij}+k^{\varepsilon}_{e,ij}+k^{\omega^2}_{e,ij}\right)\delta_{eL,j} = q_{e,i}+q^{*}_{e,i}-q^{\varepsilon}_{e,i}-q^{\omega^2}_{e,i}-m^{\varepsilon}_{e,ik}I_{kj}\varepsilon_{L,j}-m^{0}_{e,ij}\ddot{x}_{j0}, \quad (51)$$

where

$$m_{e,ij}=m_{t,ij}+m_{r,ij},\ m_{t,ij}=\int_0^L \varrho A N_{ki}N_{kj}\mathrm{d}x_1,\quad i,j=\overline{1,12},\quad k=1,2,3,$$

$$m_{r,ij}=\int_0^L \varrho I_{kl}N_{(l+3)j}N_{kj}\mathrm{d}x_1,\quad i,j=\overline{1,12},\quad k,l=1,2,3,$$

$$c_{e,ij}(\omega)=\int_0^L \varrho A\omega_{L,km}N_{mj}\mathrm{d}x_1,\ k^{\varepsilon}_{e,ij}=\int_0^L \varrho A\varepsilon_{L,km}N_{mj}\mathrm{d}x_1,$$

$$k^{\omega^2}_{e,ij}=\int_0^L \varrho A\omega_{L,km}\omega_{L,ml}N_{ki}N_{lj}\mathrm{d}x_1,\ m^{0}_{e,ij}=\int_0^L \varrho A N_{ji}\mathrm{d}x_1,\quad i=\overline{1,12},\quad j=1,2,3,$$

$$m^{\varepsilon}_{e,ij}=\int_0^L N_{j(i+3)}\mathrm{d}x_1,\ \ddot{x}_{j0}=R^T\ddot{X}_{j0},\quad m^{x}_{e,ij}=\int_0^L \varrho A N_{ji}x\mathrm{d}x,\quad i,j=1,2,3.$$

Proof We follow [14] so that we can apply the Lagrange's equations written in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\delta}_{eL,i}}\right)-\frac{\partial L}{\partial\delta_{eL,i}}=0. \quad (52)$$

Thus, using the expression (31), the motion equations are obtained (an extended presentation, see the appendix):

$$\left[\int_0^L \varrho A N_{ik}N_{jk}\mathrm{d}x_1+\int_0^L \varrho I_{kl}N^{*}_{ki}N_{lj}\mathrm{d}x\right]\ddot{\delta}_{eL,j}+2\left[\int_0^L \varrho A\omega_{L,km}N_{ki}N_{mj}\mathrm{d}x_1\right]\dot{\delta}_{eL,j} + \left[k_{e,ij}+\int_0^L \varrho A N_{ki}N_{mj}\varepsilon_{L,km}\mathrm{d}x_1+\int_0^L \varrho A\omega_{L,km}\omega_{L,ml}N_{ki}N_{lj}\mathrm{d}x_1\right]\delta_{eL,j} = q_{e,i}+q^{*}_{e,i}-q^{\varepsilon}_{e,i}-q^{\omega^2}_{e,i}-\left(\int_0^L \varrho A N_{j(i+3)}\mathrm{d}x_1\right)I_{kj}\varepsilon_{L,j}-\left(\int_0^L \varrho A N_{ji}\mathrm{d}x\right)\ddot{x}_{j0}. \quad (53)$$

Taking into account the above-mentioned notations, from (53) we obtain the desired form (51) of the motion equations. □

If the element is considered to have a constant cross section, it is possible to easily obtain the results and the coefficient after polynomial integrations.

4 Conclusions

Using the FEM to obtain the motion equations for a one-dimensional element with a general three-dimensional motion as a part of a whole multi-body system consisting of elastic elements leads us finally to obtain a system of differential equations of the form (51). Apparently, this system does not raise resolving difficulties from a theoretical point of view. But for an engineering application, where a numerical calculus is involved, some aspects of the equations can cause some difficulties.
We present some important properties of these motion equations:

- The inertia tensor m_{ij} is a symmetrical one as it happens in the case of all finite element problems in which the inertia of elements is being considered;

- The damping tensor c_{ij} is a skew-symmetric tensor and represents accelerations due to relative motions of nodal displacements with respect to the mobile reference coordinate system (Coriolis-type acceleration), linked to the moving parts of the system studied;
- The rigidity tensor k_{ij} contains both symmetric and skew-symmetric terms. Moreover, this tensor will have singularities due to the rigid motion of the system that have to be eliminated before conducting the study of the system (51);
- The vector of the generalized loads contains, beside external (concentrated and distributed) loads, terms due to inertia of finite elements being in rigid motion.

The system of differential equations obtained is nonlinear; the tensor coefficients of the system are depending on time. The method mostly used to solve such a system is that of linearizing these equations considering the tensor coefficients as being constant for very short time intervals (rigid motion freezing). In this case, a system of differential equations with constant coefficients is obtained, the resolving procedures for these equations being very well studied. Problems arise from the conservative damping caused by the skew-symmetric tensor c_{ij} and by the modification of the rigidity matrix due to relative motions.

Appendix

The derivatives of the Lagrangian:

Term in the kinetic energy	$-\frac{\partial L}{\partial \delta_{e,i}}$	$\frac{\partial L}{\partial \dot{\delta}_{e,i}}$	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\delta}_{e,i}}\right)$
$\frac{1}{2}\int_0^L \rho A \dot{X}_{ko}\dot{X}_{ko}\mathrm{d}x_l$	0	0	0
$\int_0^L \rho A \dot{X}_{ko}\dot{r}_{kl}x_l\mathrm{d}x_l$	0	0	0
$\int_0^L \rho A \dot{X}_{ko}\dot{r}_{ki}N_{ij}\delta_{eL,j}\mathrm{d}x_l$	$-\int_0^L \rho A \dot{X}_{ko}\dot{r}_{ki}N_{ij}\mathrm{d}x_l$	0	0
$\int_0^L \rho A \dot{X}_{ko}r_{ki}N_{ij}\dot{\delta}_{eL,j}\mathrm{d}x_l$	0	$\int_0^L \rho A \dot{X}_{ko}r_{ki}N_{ij}\mathrm{d}x_l$	$\int_0^L \rho A \ddot{X}_{ko}r_{ki}N_{ij}\mathrm{d}x_l$ $+\int_0^L \rho A \dot{X}_{ko}\dot{r}_{ki}N_{ij}\mathrm{d}x_l$
$\frac{1}{2}\int_0^L \rho A \dot{r}_{kl}x_l\dot{r}_{kl}x_l\mathrm{d}x_l$	0	0	0
$\int_0^L \rho A \dot{r}_{kl}x_l\dot{r}_{ki}N_{ij}\delta_{eL,j}\mathrm{d}x_l$	$-\int_0^L \rho A \dot{r}_{kl}x_l\dot{r}_{ki}N_{ij}\mathrm{d}x_l$	0	0
$\int_0^L \rho A \dot{r}_{kl}x_l r_{kl}N_{lm}\dot{\delta}_{eL,m}\mathrm{d}x_l$	0	$\int_0^L \rho A \dot{r}_{kl}x_l r_{kl}N_{lmeL,m}\mathrm{d}x_l$	$\int_0^L \rho A \ddot{r}_{kl}x_l r_{kl}N_{lmeL,m}\mathrm{d}x_l$ $+\int_0^L \rho A \dot{r}_{kl}x_l\dot{r}_{kl}N_{lmeL,m}\mathrm{d}x_l$
$\frac{1}{2}\int_0^L \rho A \dot{r}_{ki}N_{ij}\delta_{eL,j}\dot{r}_{kl}N_{im}\delta_{eL,m}\mathrm{d}x_l$	$-\frac{1}{2}\int_0^L \rho A \dot{r}_{ki}N_{ij}\delta_{eL,j}\dot{r}_{kl}N_{lm}\mathrm{d}x_l$	0	0
$\int_0^L \rho A \dot{r}_{ki}N_{ij}\delta_{eL,j}r_{kl}N_{lm}\dot{\delta}_{eL,m}\mathrm{d}x_l$	$-\int_0^L \rho A \dot{r}_{ki}N_{ij}r_{kl}N_{lm}\dot{\delta}_{eL,m}\mathrm{d}x_l$	$\int_0^L \rho A \dot{r}_{ki}N_{ij}\delta_{eL,j}r_{kl}N_{lm}\mathrm{d}x_l$	$\int_0^L \rho A \ddot{r}_{ki}N_{ij}\delta_{eL,j}r_{kl}N_{lm}\mathrm{d}x_l$ $+\int_0^L \rho A \dot{r}_{ki}N_{ij}\dot{\delta}_{eL,j}r_{kl}N_{lm}\mathrm{d}x_l$ $+\int_0^L \rho A \dot{r}_{ki}N_{ij}\delta_{eL,j}\dot{r}_{kl}N_{lm}\mathrm{d}x_l$
$\frac{1}{2}\int_0^L \rho A r_{ki}N_{ij}\dot{\delta}_{eL,j}r_{kl}N_{lm}\dot{\delta}_{eL,m}\mathrm{d}x_l$	0	$\frac{1}{2}\int_0^L \rho A r_{ki}N_{ij}\dot{\delta}_{eL,j}r_{kl}N_{lm}\mathrm{d}x_l$	$\frac{1}{2}\int_0^L \rho A \dot{r}_{ki}N_{ij}\dot{\delta}_{eL,j}r_{kl}N_{lm}\mathrm{d}x_l$ $+\frac{1}{2}\int_0^L \rho A r_{ki}N_{ij}\ddot{\delta}_{eL,j}r_{kl}N_{lm}\mathrm{d}x_l$ $+\frac{1}{2}\int_0^L \rho A r_{ki}N_{ij}\dot{\delta}_{eL,j}\dot{r}_{kl}\times N_{lm}\mathrm{d}x_l$

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