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Research Article

On Introduction to (q_1, q_2)-Linear Diophantine Fuzzy Sets and Their Applications

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The notion of parameter mappings is about creating and managing a structured relationship between parameters across different systems or processes. This concept is vital in ensuring that data remains consistent, correctly interpreted, and accurately transformed as it moves through different parts of a system or between different systems. In this paper, the concept of reference parameter mappings is introduced to handle reference parameters that will help the decision makers. To overcome the uncertainty by giving direct value to reference parameters without any rule, a new class of fuzzy sets is presented which is known as (q_1, q_2)-linear Diophantine fuzzy set $((q_1, q_2)LDFS)$, where q_1 and q_2 are reference parameter mappings. Because the q_1 and q_2 can reflect a wider variety of reference parameters than LDFSs and q-rung LDFSs, $(q_1, q_2)LDFSs$ can provide more ambiguous conditions. There is symmetry in the values of both the membership grades function and the nonmembership grades function. Furthermore, when discussing the symmetry between two or more objects, the evolution of a $((q_1, q_2)LDFSs$ via q_1 and q_2 is more adaptable than the diffused concept of a q-rung orthopair fuzzy sets or a LDFSs. The primary benefit of $(q_1, q_2)LDFSs$, which are useful in a variety of decision-making situations, is that they are able to characterize a greater number of uncertainties with respect to reference parameter mappings q_1 and q_2 than LDFSs. Next, we propose several geometric and averaging operators for a (q_1, q_2) linear Diophantine fuzzy numbers, based on established operating rules. In the latter half of the paper, different ranking algorithms based on proposed aggregation operators are presented to address a realistic assessment of the patient's high blood pressure conditions is conducted to demonstrate the viability and value of the suggested strategies.

Keywords: averaging and geometric operators; fuzzy set; MADM technique; (q_1, q_2)-linear Diophantine fuzzy set; reference parameter mappings

1. Introduction

An essential component of decision-making sciences is multiattribute decision-making ($MADM$), a procedure that can produce ranking results for the finite options based on the attribute values of various alternatives. Due to its connections to the growth of enterprises and the decision-making processes of society at large, the $MADM$ problem has drawn a lot of attention lately. One of the key challenges in decision-making is effectively and accurately conveying attribute values. Various methodologies have been developed to assist decision-makers (*DMs*) in addressing *MADM* problems. These include Fuzzy Sets (*FSs*) introduced by Zadeh [1]; Bipolar Complex Fuzzy Sets (*BCFSs*) by Mahmood et al. [2]; Cubic Intuitionistic Complex Fuzzy Soft Sets (*CIFSSs*) by Mahmood et al. [3]; Linear Diophantine Uncertain Linguistic Sets (*LDULSs*) by Mahmood et al. [4]; linear Diophantine fuzzy sets (*LDFSs*) by Yousafzai et al. [5]; and *T*-Spherical-*LDFSs* (*TSLDFS*) by Quran [6]. The *LDFS* concept, proposed by Riaz and Hashmi [7]; extends the Fuzzy Set and Intuitionistic Fuzzy Set (*IFS*) by Atanassov [8]. In this model, the satisfaction of an attribute is represented through membership, nonmembership, and reference parameters, all of which range between zero and one. In the *LDFS* framework, each element is characterized by three degrees: membership, nonmembership and reference parameters. The flexibility provided by *LDF*-Numbers (*LDFNs*) has led many researchers to explore decision-making within this context. Since their introduction, *LDFNs* have been integrated with other set theories, such as *SLDFSs*, as discussed by Alshammari et al. [9] and Farid et al. [10]. To tackle the multicriteria decision-making (*MCDM*) problem, Hashmi et al. [11] introduced *SLDFSs* and explored various properties of *SLDFSs* and *SLDFNs*, along with the spherical *LDF*-soft rough set (*SLDFSRS*) and spherical *LDF*-soft approximation space. Parimala et al. [12] proposed optimality conditions in *LDF* networks for developing a solution algorithm. Additionally, Ayub et al. [13] introduced a new concept of *LDF*-relation and further expanded the *LDFS* framework. Ayub et al. [13] introduced the concept of *LDF*-relation and expanded the framework of *LDFS*. In the same year, Iampan et al. [14] proposed several aggregation operators, including the *LDF*-Einstein weighted averaging (*LDFEWA*), *LDF*-Einstein ordered weighted averaging (*LDFEOWA*), *LDF*-Einstein weighted geometric (*LDFEWG*), and *LDF*-Einstein ordered WG (*LDFEOWG*) operators. These operators were applied to *MADM* scenarios. Additionally, Kamacı [15] introduced some algebraic properties of finite *LDF*-subsets within the contexts of group, ring, and field structures. They also introduced the concepts of *LDF*-subfields of a field, *LDF*-subrings and ideals of a ring, and *LDF*-subgroups and normal subgroups of a group. To integrate interval-valued *LDF*-information, Petchimuthu et al. [16] developed a *MCDM* approach utilizing interval-valued *LDF*WA ((*IVLDFWA*) and interval-valued *LDF*-WG aggregation (*IVLDFWGA*) operators. Additionally, Kamacı [17] introduced the complex-*LDFS* (*CLDFS*), provided illustrative examples, and explored its properties, thereby extending the generalization of *LDFSs* further. Tahan et al. [18] introduced *LDF*-subpolygroups, *LDF*-normal subpolygroups, and linear Diophantine anti-fuzzy subpolygroups of a polygroup as generalizations of fuzzy subpolygroups, fuzzy normal subpolygroups, and antifuzzy subpolygroups. They also provided examples and discussed their findings. On the other hand, the authors in [19, 20] developed new fuzzy sets that are known as Pythagorean and orthopair fuzzy sets. Additionally, Mohammad et al. [21] developed a multicriteria group decision-making (*MCGDM*) process based on a similarity measure using *LDF*-information. Prakash et al. [22] introduced specific forms of *LDF*-structures, including bridges, cut-vertices, cycles, trees, and forests. Meanwhile, Almagrabi et al. [23] extended the classical arithmetic and geometric averages to the q-rung *LDF*-framework. They developed two *q*-rung *LDF*-operators based on the fundamental operations and comparison methods for q-rung *LDF*Ns (*q-RLDFNs*). The *q*-rung *LDFS* (*q-RLDFS*) encompasses the membership degree and nonmembership degree, ensuring that the sum of the qth power of the membership, nonmembership, and reference parameters does not exceed one. The *q-RLDFS* provides greater flexibility in expressing *LDF*-information through its membership degrees. In addition, the researchers proposed the *q*-rung *LDF*WGA (*q-RLDFWGA*) (*q-RLDFWGA*), *q-RLDFOWGA*, and *q-RLDFHWGA* operators. Ashraf et al. [24] further advanced the field by introducing the spherical-*LDF* (*Sq-RLDFS*), a novel generalization that integrates the *LDFS*, *q-RLDFS*, and spherical-*LDF* (*SLDFS*), highlighting its key features and potential applications. Later on, the idea of trapezoidal *LDF*Ns (Trap. *LDFNs*) was presented by Iqbal and Yaqoob [25]. After that [26], introduced the (p, q)-rung LDFSs with application in decision-making field. Asif et al. [27]; Wang et al. [28]; Mahmood et al. [29]; and Xu et al. [30] introduced different types of aggregation operators in various fuzzy environments. Furthermore, in order to handle *LDF*-information, Riaz et al. [31] created some *MCDM* procedure that makes use of*LDFWA* and*LDFWG* operators. Under certain real world problem, the sum of membership grades (*MG*) and nonmembership grades (*NMG*) to in any types of *FS* is sometimes greater than 1 for example $0.9 + 0.7 > 1$ and the square sum may also be greater than 1 (e.g., $(0.9)^2 + (0.7)^2 > 1$). In such cases, *IFS* and *PyFS* have failed. In the case of *q-ROFS*, the conditions on *MG* and *NMG* are modified to $0 \leq \varsigma(x)^q + \psi(x)^q \leq 1$, in order to overcome these deficiencies. We can handle *MG* and *NMG* even for extremely large values. In certain practical issues we obtain $1^q + 1^q > 1$, which violates the restriction of *q-ROFS*, if both *MG* $\psi(x)$ and *NMG* $\varsigma(x)$ are equal to 1 (i.e., $\varsigma(x) = \psi(x) = 1$). Then [7], introduced concept of *LDFS* in which they introduced the role of reference parameters, which hold the condition $0 \leq \alpha\varsigma(x) + \beta\psi(x) \leq 1$ with $0 \leq \alpha + \beta \leq 1$. Here, the weak point of LDFS is to give the direct value of reference parameters so due to this reason, this is causing an increase in uncertainty e.g., if parameter $\alpha = \text{cheap}$ then parameter, $\beta = \text{expensive}$, then $\text{cheap}, \text{expensive} \in [0, 1]$ so this is not the right way to introduce the values of these parameters cheap and expensive. The suitable way is to introduce the proper criteria's (q_1, q_2) that give values of theses parameters such that $q_1\colon J_1 = \{\alpha = \text{cheap}\} \longrightarrow [0, 1]$ and $q_2\colon J_2 = \{\beta = \text{expensive}\} \longrightarrow [0, 1]$ which are known as reference parameter mappings. In order to eliminate this contradiction, we introduce the novel idea of the (q_1, q_2)-LDFSs which is capable of dealing with these situations. To explain the concept of (q_1, q_2)-LDFSs, we have five objectives related to our proposed method. Due to the usage of reference parameter mappings $q_1: J_1 \longrightarrow [0, 1]$ and $q_2: J_2 = \longrightarrow [0, 1]$, the suggested decision-making model of the (q_1, q_2)-*LDFN* is more flexible and efficient than existing techniques. In *MADM* situations, the (q_1, q_2)-*LDFS* also classifies the input by altering the physical sense of reference parameters with respect to pair of mappings (q_1, q_2). This article discusses the sources of inspiration for the proposed work in every section. This paper aims to accomplish the following several goals:

a. To define a new fuzzy set that is (q_1, q_2)-LDFSs.
b. To specify a few characteristics of (q_1, q_2)-LDFSs with respect to reference parameter mappings q_1 and q_2.
c. To provide a novel score functions that may be applied to (q_1, q_2)-*LDFMADM*.
d. To define and investigate some of the properties of a new class of aggregation operators, namely the (q_1, q_2)-LDF weighted averaging ((q_1, q_2)-*LDFWAA*) and (q_1, q_2)-*LDFWGA* operators.
e. To create optimization (q_1, q_2)-*LDFMADM* models in order to ascertain the attribute weights.

Here is the brief introduction of "reference parameter mappings."

"Reference parameter mappings" typically refer to the mappings used to link or reference parameters between different entities, systems, or contexts. This concept is often used in scenarios where multiple systems or processes need to work together, and parameters or variables from one system must be correctly referenced in another.

Here's how "reference parameter mappings" are used in various contexts:

The reference parameter mappings extend the concept of reference parameters by establishing structured relationships between parameters across different contexts, systems, or decision criteria. Mappings accommodate varying conditions and criteria that affect reference parameters, maintaining the model's flexibility and accuracy even when inputs or constraints evolve. In short, reference parameter mappings expand the utility of reference parameters by capturing complex relationships and dependencies, ensuring consistency, adaptability, and precision across contexts. Reference parameter mappings in decision-making involve the systematic alignment of key decision variables with specific parameters to structure the decision-making process. This technique enhances clarity by linking inputs such as costs, benefits, risks, or performance indicators to the corresponding decision criteria or objectives. By mapping these relationships, it becomes possible to analyze the impact of each parameter on outcomes, assign appropriate weights to factors, and optimize decisions. Advantages of reference parameter mappings include improved decision accuracy, consistency, and transparency. They enable better prioritization of resources, support the development of predictive models, and enhance communication among stakeholders by illustrating the rationale behind choices. Applications of this concept are found across various domains, including business strategy (aligning costs and revenues to financial goals), policy formulation (linking economic indicators to societal objectives), project management (mapping milestones to resource allocation), and healthcare (connecting treatment plans to patient outcomes).

For example, imagine a scenario where an e-commerce platform (System A) needs to integrate with a payment gateway (System B). System A might have a parameter order ID, while System B expects a transactionID. Reference parameter mappings would ensure that when System A sends data to System B, the orderID is correctly mapped to transactionID, allowing the payment process to proceed smoothly.

The concept of parameter mappings is defined in various direction of mathematics like convex theory, differential equations, partial differential equations etc. For instance, Varošanec [32] introduced the concept of $\hbar$-convexity such that let K be convex set and parameter mapping $\hbar: [0, 1] \subseteq K \longrightarrow \mathfrak{R}^+$ such that $\hbar \not\equiv 0$. Then, a mapping $\mathscr{P}: K \longrightarrow \mathfrak{R}$ is said to be $\hbar$-convex function on K if $\mathscr{P}(\alpha б + (1-\alpha)s) \leq \hbar(\alpha)\mathscr{P}(б) + \hbar(1-\alpha)\mathscr{P}(s)$, for all $б, s \in K$, $\alpha \in [0, 1]$. Similarly, Khan et al. [33] initiated to propose the idea of $\hbar$-convex fuzzy-number valued mappings such that let K be convex set and parameter mapping $\hbar: [0, 1] \subseteq K \longrightarrow \mathfrak{R}^+$ such that $\hbar \not\equiv 0$. Then, a mapping $\widetilde{\mathscr{P}}: K \longrightarrow \mathscr{F}_c$ is said to be $\hbar$-convex fuzzy-valued function on K if $\widetilde{\mathscr{P}}(\alpha б + (1-\alpha)s) \leq_{\mathbb{F}} \hbar(\alpha)\widetilde{\mathscr{P}}(б) \oplus \hbar(1-\alpha)\widetilde{\mathscr{P}}(s)$, for all $б, s \in K$, $\alpha \in [0, 1]$, where $\widetilde{\mathscr{P}}(б) \geq_{\mathbb{F}} \widetilde{0}$ and $\mathscr{F}_c$ is a set of fuzzy number.

Inspired by the ongoing approach, a new concept has been introduced in fuzzy theory. In order to accomplish these objectives, the primary framework of this document is presented below. In Section 2, the basic notions of *LDFS* and generalized *LDFSs* are presented. In Section 3, we develop a certain class of new *FSs* that is known as (q_1, q_2)-*LDFSs*. Additionally, analysis over (q_1, q_2)-*LDFSs* is also given as well as interpretation of sensitivity and comparison is also discussed. Some basic operations and relations are also proved. We build the (q_1, q_2)-*LDFWAA*, (q_1, q_2)-*LDFOWAA*, and (q_1, q_2)-*LDFHWAA* operators in Section 4, and we thoroughly investigate several well-known and practicable properties and exceptional outcomes. In Section 5, We present the (q_1, q_2)-*LDFWGA*, (q_1, q_2)- *LDFOWGA*, and (q_1, q_2)-*LDFHWGA* operators and also characterize the properties of these operators. In Section 6, we suggest MADM techniques for assessing green supply chain management in the context of the (q_1, q_2)-LDF information. The comparison analysis highlights the benefits of these techniques. Section 7 provides an explanation of the paper's closing remarks.

2. Background Information and Notation

In this section, we first go over the fundamental idea and its understanding-related features before developing a new one. Now, we start with the basic definition of intuitionistic fuzzy set such that.

Definition 1. Zadeh [1] suppose an arbitrary nonempty set E. A fuzzy set $\mathscr{L}$ is defined on E as

$$\mathscr{L} = \{(x, \varsigma(x)) | x \in E\}. \tag{1}$$

Here, the function ς is a transformation of E to $[0, 1]$, and for every $x \in E, 0 \leq \varsigma(x) \leq 1$, and function $\varsigma(x)$ are said to be the membership grade (MG) of x in E.

Definition 2 (see [8]). Let us have a fixed universe E and its sub-set $\mathscr{L}$. The set

$$\mathscr{L} = \{\langle x, \varsigma(x), \psi(x)\rangle : \text{for all } x \in E\}, \tag{2}$$

where $0 \leq \varsigma(x) + \psi(x) \leq 1$, is called intuitionistic fuzzy set and functions $\varsigma, \psi: E \longrightarrow [0, 1]$ indicate the degree of membership (MG) (validity, etc.) and nonmembership grades (NMG) (nonvalidity, etc.) of element $x \in E$ to a fixed set $\mathscr{L} \subseteq E$. Now, we can define also function $\pi: E \longrightarrow [0, 1]$ by means of

$$\pi(x) = 1 - \varsigma(x) - \psi(x). \tag{3}$$

and it corresponds to degree of indeterminacy (uncertainty, etc.), see Figure 1.

Definition 3 (see [19]). Consider a fixed set E and the Pythagorean fuzzy set (PyFS) is denoted by A_P and mathematical defined as

$$A_P = \{(x, \varsigma(x), \psi(x)) | x \in E\}, \tag{4}$$

where $\varsigma(x)$ and $\psi(x) \in [0, 1]$ are MG and NMG functions with subject to $0 \leq (\varsigma(x))^2 + (\psi(x))^2 \leq 1$, see Figure 1. The hesitancy MG is denoted by

$$\pi(x) = \sqrt{1 - (\varsigma(x))^2 - (\psi(x))^2}. \tag{5}$$

Definition 4 (see [20]). Suppose E be a fixed set. A q-rung orthopair fuzzy set (q-ROFS) B on E have the following mathematical symbol;

$$B = \{(x, \varsigma(x), \psi(x)) : x \in E\}, \tag{6}$$

where $\varsigma(x)$ and $\psi(x) \in [0, 1]$ are MG and NMG functions with subject to $0 \leq (\varsigma(x))^q + (\psi(x))^q \leq 1; q \geq 1$, see Figure 1. The hesitancy part is denoted as

$$\pi(x) = \sqrt[q]{1 - (\varsigma(x))^q - (\psi(x))^q}, \tag{7}$$

see Figure 1, see [20].

Note that, it is not possible to introduce $q \longrightarrow \infty$ because it is not easy to give this value for decision makers.

Definition 5 (see [7]). Suppose E be a fixed nonempty reference set and the LDFS is denoted by G_D and mathematical defined as

$$G_D = \{(x, \langle \varsigma(x), \psi(x)\rangle, \langle \alpha, \beta\rangle) : x \in E\}, \tag{8}$$

where $\varsigma(x), \psi(x), \alpha, \beta \in [0, 1]$ are MG, NMG and references parameters (RPs), respectively, and hold the condition

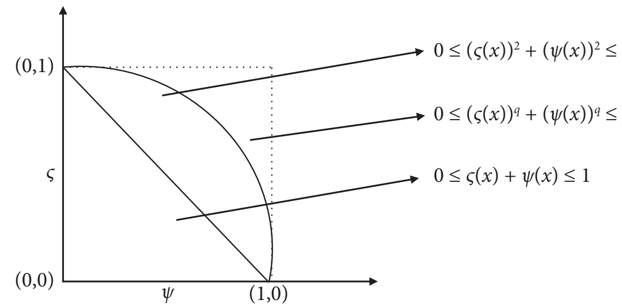

Figure 1: A comparison between intuitionistic fuzzy space, pythagorean fuzzy space and q-rung orthopair fuzzy space.

$0 \leq \alpha\varsigma(x) + \beta\psi(x) \leq 1, \forall x \in E$ with $0 \leq \alpha + \beta \leq 1$. Such reference parameters may help to describe or identify a specific model. Indeterminacy degree can be defined as

$$\gamma\pi(x) = 1 - (\alpha)\varsigma(x) - (\beta)\psi(x), \tag{9}$$

where γ is the reference parameter of the indeterminacy degree.

3. The (q_1, q_2)-Fuzzy Set

In this sections, we firstly start with the main definition of (q_1, q_2)-LDFSs such that:

The "reference parameter mappings" extend the concept of "reference parameters" by establishing structured relationships between parameters across different contexts, systems, or decision criteria. Mappings accommodate varying conditions and criteria that affect reference parameters, maintaining the model's flexibility and accuracy even when inputs or constraints evolve. In short, reference parameter mappings expand the utility of reference parameters by capturing complex relationships and dependencies, ensuring consistency, adaptability, and precision across contexts. Reference parameter mappings in decision-making involve the systematic alignment of key decision variables with specific parameters to structure the decision-making process. This technique enhances clarity by linking inputs such as costs, benefits, risks, or performance indicators to the corresponding decision criteria or objectives. By mapping these relationships, it becomes possible to analyze the impact of each parameter on outcomes, assign appropriate weights to factors, and optimize decisions. Advantages of reference parameter mappings include improved decision accuracy, consistency, and transparency. They enable better prioritization of resources, support the development of predictive models, and enhance communication among stakeholders by illustrating the rationale behind choices. Applications of this concept are found across various domains, including business strategy (aligning costs and revenues to financial goals), policy formulation (linking economic indicators to societal objectives), project management (mapping milestones to resource allocation), and healthcare (connecting treatment plans to patient outcomes).

Definition 6. Let us have a fixed universe E and its sub-set T. The set

$$T = \{(x, \langle \varsubsetneq (x), s(x)\rangle, \langle q_1(\alpha), q_2(\beta)\rangle): \text{for all } x \in E\}, \tag{10}$$

where $0 \leq q_1(\alpha)\varsubsetneq(x) + q_2(\beta)\psi(x) \leq 1$, with $0 \leq q_1(\alpha) + q_2(\beta) \leq 1$, is called (q_1, q_2)-LDFS ((q_1, q_2)-*LDFS*) and functions $\varsubsetneq, \psi: E \longrightarrow [0, 1]$ indicate the degree of membership (validity, etc.) and nonmembership (nonvalidity, etc.) of element $x \in E$ to a fixed set $T \subseteq E$ as well as $q_1: J_1 \longrightarrow [0, 1]$, and $q_2: J_2 \longrightarrow [0, 1]$ are parameter mappings which are known as reference parameters mappings, where $\alpha \in J_1$, and $\beta \in J_2$,. Now, we can define also function $\pi: E \longrightarrow [0, 1]$ by means of

$$q_3(\gamma)\pi(x) = 1 - q_1(\alpha)\varsubsetneq(x) - q_2(\beta)s(x), \tag{11}$$

and it corresponds to degree of indeterminacy (uncertainty, etc.), where $q_3: J_3 \longrightarrow [0, 1]$ and $\gamma \in J_3$, see Figure 2.

For $1 \geq q_1(\alpha), q_2(\beta) \geq 0$, the complete square with vertices $((\varsubsetneq, \psi), (0, 0))$, $((\varsubsetneq, \psi), (0, 1))$, $((\varsubsetneq, \psi), (1, 1))$, and $((\varsubsetneq, \psi), (1, 0))$ is formed. Consequently, the membership and nonmembership values of (q_1, q_2)-*LDFS* are not influenced by variations in $q_1(\alpha)$ and $q_2(\beta)$. All points within the square that satisfy the condition $0 \leq q_1(\alpha)\varsubsetneq(x) + q_2(\beta)\psi(x) \leq 1$ are represented in Figure 2. It is notable that the light gray shading encompasses the entire region inside the square.

Point satisfying $0 \leq q_1(\alpha)\varsubsetneq(x) + q_2(\beta)\psi(x) \leq 1$.

Note that if J_1, J_2, and J_3 are sets of reference parameters, then the q_1, q_3, and q_3 are representing the special criteria to choose suitable values for reference parameters. Additionally, one can define the criteria's through the following way such that q_1, q_2, q_3: $J \longrightarrow [0, 1]$, where $J = J_1 \cup J_2 \cup J_3$.

Remark 1. Here some novel and classical exceptional cases are acquired such that

1. If $q_1 = q_2 = q$, then (q_1, q_2)-*LDFS* reduces to q-*LDFS*.
2. Let J_1, J_2 = set of RPs and $\in J_1$, $\beta \in J_2$. If $q_1(\alpha) = \alpha$ and $q_2(\beta) = \beta$, then one can acquired the definition of *LDFS*.
3. Let J_1, J_2 = set of RPs and $\alpha, \beta \in J$. If $q_1(\alpha) = \alpha^q$ and $q_2(\beta) = \beta^q$ and $q \geq 1$, then one can acquired the definition of q-*RLDFS*.
4. If $J = \mathbb{R}\backslash[0, 2)$ and $q_1(q) = q_2(q) = 1/q$, where $p, q \in J$, then one can get definition of q-fractional fuzzy set.
5. Let J_1, J_2 = set of RPs and $\alpha, \beta \in J$. If $q_1(\alpha) = \gamma$ and $q_2(\beta) = \delta$ with $\gamma, \delta \in [0, 1]$, then one can acquired the new version of *LDFS*. This approach is strong as compare to Remark 2 and 3.
6. Let J_1, J_2 be two set of RPs and $\alpha, \beta \in J$. If $q_1(\alpha) = \gamma^q$ and $q_2(\beta) = \delta^q$ with $\gamma, \delta \in [0, 1]$ and $q \geq 1$, then one can acquired the new definition of q-*RLDFS*.

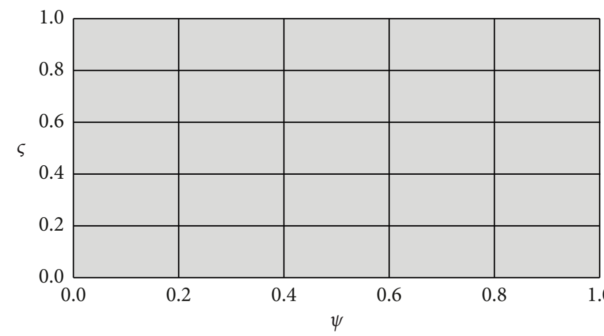

Figure 2: Graphical representation of (q_1, q_2)-linear diophantine fuzzy set.

7. If $J = \mathbb{R}^+\backslash[0, 2)$ and $q_1(p) = (1/p), q_2(q) = 1/q$, where $q \in J$, then one can get the another new case which is known as (p, q)-fractional fuzzy set.
8. From Remark 7, if $p = q$, then one can achieved the definition of q-fractional fuzzy set.
9. By using Remarks (2) and (4) approaches, if someone takes $q_1(\alpha) = \alpha/q$ and $q_2(\beta) = \beta/q$, with $q \geq 2$, then we have another version of (q_1, q_2)-*LDFS* which is known as q-fractional LDFS (q-*FLDFS*).
10. By using Remarks (3) and (4) approaches, then one can take $q_1(\alpha) = \alpha^q/(1 + q)$ and $q_2(\beta) = \beta^q/(1 + q)$, with $q \geq 1$, then we have new version of (q_1, q_2)-*LDFS*.
11. With the help of Remarks (4) and (5), if someone takes $q_1(\alpha) = \alpha/q$ and $q_2(\beta) = \beta/q$, with $q \geq 2$, and with $\gamma, \delta \in [0, 1]$, then we have another version of (q_1, q_2)-*LDFS*.
12. With the support of Remarks (4) and (6), then one can take $q_1(\alpha) = \gamma^q/(1 + q)$ and $q_2(\beta) = \delta^q/(1 + q)$ with $\gamma, \delta \in [0, 1]$ and $q \geq 1$, then we have new version of (q_1, q_2)-*LDFS*.
13. Let J = set of RPs and $\alpha, \beta \in J$. If $q_1(\alpha) = \alpha^p$ and $q_2(\beta) = \beta^q$ and $q \geq 1$, then one can acquired the definition of (p, q)-*RLDFS*.

Similarly, the new exceptional cases have been obtained like Remarks (1)–(12). It means that (q_1, q_2)-*LDFS* is best outcome in field of fuzzy theory as compare to other classical fuzzy sets like *IFS*, *LDFS*, -*RLDFS*, q-rung orthopair fuzzy sets and q-fractional fuzzy sets, etc.

Definition 7. A (q_1, q_2)-LDFN is a collection of

$$\Upsilon = \{\langle \varsubsetneq, \psi\rangle, \langle q_1, q_2\rangle\}, \tag{12}$$

where Υ represent the (q_1, q_2)-LDFN with conditions;

i. $0 \leq q_1(\alpha) + q_2(\beta) \leq 1$,
ii. $0 \leq q_1(\alpha)\varsubsetneq(x) + q_2(\beta)\psi(x) \leq 1$
iii. $q_1(\alpha), \varsubsetneq(x), q_2(\beta), \psi(x) \in [0, 1]$, for all $x \in E$, $\alpha \in J_1$, and $\beta \in J_2$.

For the sake of simplicity, the set of (q_1, q_2)-LDFN ((q_1, q_2)-*LDFNs*).

Next definition is about absolute (q_1, q_2)- LDFSs and null or empty (q_1, q_2)-LDFSs.

Definition 8. A (q_1, q_2)-LDFSs on E of the form

$$^{1}\Upsilon = \{(x, (1, 0), (1, 0)): x \in E\}, \tag{13}$$

is called absolute (q_1, q_2)-LDFSs and

$^{0}\Upsilon = \{(x, (0, 1), (0, 1)): x \in E\}$ is called empty or null (q_1, q_2)-LDFSs.

It is noteworthy to notice that these definitions generalize one given in [Riaz and Hashim] for absolute and null (q_1, q_2)-LDFSs.

It is important to remember that the (q_1, q_2)-LDF space grows as the due to q_1 and q_2. As a result, the boundary limits have a larger search space that can represent a wider range of the fuzzy data.

3.1. The Receptionist Selection for Hotel. The (q_1, q_2)-*LDFSs* finds several real-world applications in the domains of MADAM, engineering, AI, and medical fields, and mass assembly and distribution. There is a broad range of various implementations in this novel concept. Assume that a receptionist is required for the hotel. The hotel management hiring this post under the following pair of RPs (q_1, q_2) based on the following qualities: reliable, time management, attention to deal, respecting, written communication, verbal communication and cooperating-all of which were necessary for choosing an ideal receptionist with a wide range of qualities at a reasonable pay. Assume that the collection $\Upsilon = \{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5\}$ represents the receptionist. First, assume that the (q_1, q_2)-*LDFS* structure has reference parameters (α, β) with (q_1, q_2). Let α represent mental reliable, time management, attention to deal, respecting, and β represent written communication, verbal communication, cooperating. Table 1 shows the (q_1, q_2)-*LDFSs* of these parameters in tabular form.

For the second group under (q_1, q_2), we can take into consideration α = good experience and β = not good experience then $q_1(\alpha)$ is increases and $q_2(\beta)$ decreases due to the physical sense of experience time. The tabular form of (q_1, q_2)-*LDFS* for the second group reference parameters is shown in Table 2. The data can now be categorized as (q_1, q_2)-*LDFS* in different ways if we wish to change the physical meaning of these parameters.

3.2. Basic Operations on (q_1, q_2)-Fuzzy Sets. This section proposed some of the basic operations on (q_1, q_2)-LDFSs like inclusion, union, intersection, complement, and some compositions as well as some properties are also illustrated. For the sake of easy understanding, we will take the following three (q_1, q_2)-LDFSs over fixed universe E:

Table 1: (q_1, q_2)-*LDFS* via first group of (q_1, q_2).

Υ	$(\langle \varsigma, \psi \rangle, \langle q_1, q_2 \rangle)$
$\mathfrak{R}_1$	$(\langle 1, 0.959 \rangle, \langle 0.498, 0.333 \rangle)$
$\mathfrak{R}_2$	$(\langle 1, 0.931 \rangle, \langle 0.7, 0.1172 \rangle)$
$\mathfrak{R}_3$	$(\langle 0.864, 0.909 \rangle, \langle 0.672, 0.245 \rangle)$
$\mathfrak{R}_4$	$(\langle 1, 0.966 \rangle, \langle 0.803, 0.134 \rangle)$
$\mathfrak{R}_5$	$(\langle 1, 0.966 \rangle, \langle 0.403, 0.534 \rangle)$

Table 2: (q_1, q_2)-*LDFS* via second group of (q_1, q_2).

Υ	$(\langle \varsigma, \psi \rangle, \langle q_1, q_2 \rangle)$
$\mathfrak{R}_1$	$(\langle 0.7, 0.8 \rangle, \langle 0.3, 0.4 \rangle)$
$\mathfrak{R}_2$	$(\langle 0.8, 0.9 \rangle, \langle 0.2, 0.7 \rangle)$
$\mathfrak{R}_3$	$(\langle 0.8, 0.9 \rangle, \langle 0.3, 0.6 \rangle)$
$\mathfrak{R}_4$	$(\langle 0.6, 0.9 \rangle, \langle 0.3, 0.6 \rangle)$
$\mathfrak{R}_5$	$(\langle 0.5, 0.8 \rangle, \langle 0.4, 0.5 \rangle)$

$$\begin{aligned}
\Upsilon &= \{(x, \langle \varsigma_\Upsilon(x), \psi_\Upsilon(x) \rangle, \langle q_{1\Upsilon}(\alpha), q_{2\Upsilon}(\beta) \rangle): \text{for all } x \in E\},\\
Y &= \{(x, \langle \varsigma_Y(x), \psi_Y(x) \rangle, \langle q_{1Y}(\alpha), q_{2Y}(\beta) \rangle): \text{for all } x \in E\},\\
Z &= \{(x, \langle \varsigma_Z(x), \psi_Z(x) \rangle, \langle q_{1Z}(\alpha), q_{2Z}(\beta) \rangle): \text{for all } x \in E\}.
\end{aligned} \tag{14}$$

Definition 9. Let Υ and Y be two (q_1, q_2)-LDFSs. Then,

- $\Upsilon \subseteq Y$ iff $\varsigma_\Upsilon(x) \preccurlyeq \varsigma_Y(x)$, $\psi_\Upsilon(x) \succcurlyeq \psi_Y(x)$, $q_{1\Upsilon}(\alpha) \preccurlyeq q_{1Y}(\alpha)$ and $q_{2\Upsilon}(\beta) \succcurlyeq q_{2Y}(\beta)$,
- $\Upsilon = Y$ iff $\Upsilon \subseteq Y$ and $\Upsilon \supseteq Y$,
- $\Upsilon \cup Y = \{(x, \langle \curlyvee(\varsigma_\Upsilon(x), \varsigma_Y(x)), \curlywedge(\psi_\Upsilon(x), \psi_Y(x)) \rangle, \langle \curlyvee(q_{1\Upsilon}(\alpha), q_{1Y}(\alpha)), \curlywedge(q_{2\Upsilon}(\beta), q_{2Y}(\beta)) \rangle) : \text{for all } x \in E\}$,
- $\Upsilon \cap Y = \{(x, \langle \curlywedge(\varsigma_\Upsilon(x), \varsigma_Y(x)), \curlyvee(\psi_\Upsilon(x), \psi_Y(x)) \rangle, \langle \curlywedge(q_{1\Upsilon}(\alpha), q_{1Y}(\alpha)), \curlyvee(q_{2\Upsilon}(\beta), q_{2Y}(\beta)) \rangle) : \text{for all } x \in E\}$,
- $\Upsilon^c = \{(x, \langle \psi_\Upsilon(x), \varsigma_\Upsilon(x) \rangle, \langle q_{2\Upsilon}(\beta), q_{1\Upsilon}(\alpha) \rangle) : \text{for all } x \in E\}$.

Proposition 1. *Let Υ, Y and Z be three (q_1, q_2)-LDFSs. Then, following properties holds such that*

1. *$\Upsilon \subseteq Y$ and $Y \subseteq Z$ implies $\Upsilon \subseteq Z$; (Inclusion property),*
2. *$\Upsilon \cup Y = Y \cup \Upsilon$ and $\Upsilon \cap Y = Y \cap \Upsilon$; (Commutative law),*
3. *$\Upsilon \cup (Y \cup Z) = (\Upsilon \cup Y) \cup Z$ and $\Upsilon \cap (Y \cap Z) = (\Upsilon \cap Y) \cap Z$; (Associative law)*
4. *$\Upsilon \cup (Y \cap Z) = (\Upsilon \cup Y) \cap (\Upsilon \cup Z)$ and $\Upsilon \cap (Y \cup Z) = (\Upsilon \cap Y) \cup (\Upsilon \cap Z)$; (Distributive laws)*
5. *De-Morgan's Laws holds for Υ and Y.*

Proof 1. (1) Consider $\Upsilon \subseteq Y$ and $Y \subseteq Z$, then by Definition 2, we have

$$\begin{aligned}
&\varsigma_{\Upsilon}(x) \preccurlyeq \varsigma_{Y}(x), \\
&\psi_{\Upsilon}(x) \succcurlyeq \psi_{Y}(x), \\
&q_{1\Upsilon}(\alpha) \preccurlyeq q_{1Y}(\alpha), \\
&q_{2\Upsilon}(\beta) \succcurlyeq q_{2Y}(\beta),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\varsigma_{Y}(x) \preccurlyeq \varsigma_{Z}(x), \\
&\psi_{Y}(x) \succcurlyeq \psi_{Z}(x), \\
&q_{1Y}(\alpha) \preccurlyeq q_{1Z}(\alpha), \\
&q_{2Y}(\beta) \succcurlyeq q_{2Z}(\beta).
\end{aligned} \tag{16}$$

Combining (15) and (16), we have

$$\begin{aligned}
&\varsigma_{\Upsilon}(x) \preccurlyeq \varsigma_{Y}(x) \preccurlyeq \varsigma_{Z}(x), \\
&\psi_{\Upsilon}(x) \succcurlyeq \psi_{Y}(x) \succcurlyeq \psi_{Z}(x), \\
&q_{1\Upsilon}(\alpha) \preccurlyeq q_{1Y}(\alpha) \preccurlyeq q_{1Z}(\alpha), \\
&q_{2\Upsilon}(\beta) \succcurlyeq q_{2Y}(\beta) \succcurlyeq q_{2Z}(\beta).
\end{aligned} \tag{17}$$

From (17), we conclude that

$$\begin{aligned}
&\varsigma_{\Upsilon}(x) \preccurlyeq \varsigma_{Z}(x), \\
&\psi_{\Upsilon}(x) \succcurlyeq \psi_{Z}(x), \\
&q_{1\Upsilon}(\alpha) \preccurlyeq q_{1Z}(\alpha), \\
&q_{2\Upsilon}(\beta) \succcurlyeq q_{2Z}(\beta).
\end{aligned} \tag{18}$$

Hence, $\Upsilon \subseteq Z$.

Similarly, the remailing results 2–5 can be proved easily. □

3.3. Comparison Between Two (q_1, q_2)-LDFSs. It is well known fact that comparison laws in fuzzy theory play a critical role, especially in the field of decision making and some other optimization problems. These laws enable us to differentiate the two (q_1, q_2)-LDFSs as well as sometime these rules tell us the worth of the relation between these two (q_1, q_2)-LDFSs that this relation is how much strong.

Definition 10. Let $\Upsilon = \{\langle \varsigma_{\Upsilon}, \psi_{\Upsilon}\rangle, \langle q_{1\Upsilon}, q_{2\Upsilon}\rangle\}$ be a (q_1, q_2)-*LDFN*. Then, score function $(S_{(q_1,q_2)\text{-}LDFN}(\Upsilon))$ and accuracy functions $(H_{(q_1,q_2)\text{-}LDFN}(\Upsilon))$ of Υ are denoted and defined as

$$S_{(q_1,q_2)\text{-}LDFN}(\Upsilon) = \frac{1}{2}\left[\varsigma_{\Upsilon} - \psi_{\Upsilon} + q_1 - q_2\right], \tag{19}$$

where $-1 \leq S_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \leq 1$.

$$H_{(q_1,q_2)\text{-}LDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\varsigma_{\Upsilon} + \psi_{\Upsilon}}{2} + q_1 + q_2\right], \tag{20}$$

where $0 \leq H_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \leq 1$, respectively. These rules define the comparison between two (q_1, q_2)-*LDFNs* Υ_1 and Υ_2 such that

a. Υ_1 is higher ranked than Υ_2 if $S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) > S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

b. Υ_1 is lower ranked than Υ_2 if $S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) < S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

When $S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) = S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$ for two (q_1, q_2)-*LDFNs*, then

c. Υ_1 is higher ranked than Υ_2 if $H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) > H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

d. Υ_1 is lower ranked than Υ_2 if $H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) < H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

e. Υ_1 is similar Υ_2 if $H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) = H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

Example 1. Let $\Upsilon_1 = (\langle 1, 0.9\rangle, \langle 0.3, 0.3\rangle)$ and $\Upsilon_2 = (\langle 0.4, 0.9\rangle, \langle 0.5, 0.5\rangle)$ be two alternative with (q_1, q_2)-*LDFNs*. Then, score function is utilized to determine the preferred option such that

$$\begin{aligned}
S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) &= \frac{1}{2}[1 - 0.9 + 0] = 0.05, \\
S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2) &= \frac{1}{2}[4 - 0.9 + 0] = -0.2,
\end{aligned} \tag{21}$$

hence, option Υ_2 is preferable to option Υ_1.

Example 2. If (q_1, q_2)-*LDFNs* for two alternative are $\Upsilon_2 = (\langle 1, 0.5\rangle, \langle 0.4, 0.4\rangle)$ and $\Upsilon_2 = (\langle 0.9, 0.4\rangle, \langle 0.5, 0.5\rangle)$, then score function is utilized to determine the preferred option such that

$$\begin{aligned}
S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) &= \frac{1 - 0.5}{2} = 0.25, \\
S_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2) &= \frac{0.9 - 0.4}{2} = 0.25.
\end{aligned} \tag{22}$$

So, we are unsure of which option is preferable in this situation. However, by using (20), we can get

$$\begin{aligned}
H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) &= \frac{1 + 0.5}{4} + \frac{0.4 + 0.4}{2} = 0.78, \\
H_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2) &= \frac{0.9 + 0.4}{4} + \frac{0.5 + 0.5}{2} = 0.83.
\end{aligned} \tag{23}$$

As a result, alternative Υ_1 is superior to alternative Υ_2.

The quadratic score function for (q_1, q_2)-*LDFN* is defined in the next subsection.

Definition 11. Let $\Upsilon = \{\langle \varsigma_{\Upsilon}, \psi_{\Upsilon}\rangle, \langle q_{1\Upsilon}, q_{2\Upsilon}\rangle\}$ be a (q_1, q_2)-*LDFN*. Then, quadratic score function $(\$_{(q_1,q_2)\text{-}LDFN}(\Upsilon))$ and quadratic accuracy functions $(И_{(q_1,q_2)\text{-}LDFN}(\Upsilon))$ of Υ are denoted and defined as

$$\$_{(q_1,q_2)\text{-}LDFN}(\Upsilon) = \frac{1}{2}\left[\varsigma_{\Upsilon}^{\,2} - \psi_{\Upsilon}^{\,2} + q_1^{\,2} - q_2^{\,2}\right], \tag{24}$$

where $-1 \leq \$_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \leq 1$.

$$И_{(q_1,q_2)\text{-}LDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\varsigma_{\Upsilon}^{\,2} + \psi_{\Upsilon}^{\,2}}{2} + q_1^{\,2} + q_2^{\,2}\right], \tag{25}$$

where $0 \leq И_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \leq 1$, respectively. These rules define the comparison between two (q_1,q_2)-*LDFNs* Υ_1 and Υ_2 such that

i. Υ_1 is higher ranked than Υ_2 if $\$_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) > \$_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.
ii. Υ_1 is lower ranked than Υ_2 if $\$_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) < \$_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

When $\$_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) = \$_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$ for two (q_1,q_2)-*LDFNs*, then

iii. Υ_1 is higher ranked than Υ_2 if $И_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) > И_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.
iv. Υ_1 is lower ranked than Υ_2 if $И_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) < И_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

Υ_1 is similar Υ_2 if $И_{(q_1,q_2)\text{-}LDFN}(\Upsilon_1) = И_{(q_1,q_2)\text{-}LDFN}(\Upsilon_2)$.

Next, we introduce the expectation score function (*ESF*), which is an additional generalized score function.

Definition 12. Let $\Upsilon = \{\langle \varsigma_\Upsilon, \psi_\Upsilon\rangle, \langle q_{1\Upsilon}, q_{2\Upsilon}\rangle\}$ be a (q_1,q_2)-*LDFN*. Then, *ESF* ($\check{E}_{(q_1,q_2)\text{-}LDFN}(\Upsilon)$) of Υ are denoted and defined as

$$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\varsigma_\Upsilon - \psi_\Upsilon + 1}{2} + \frac{q_1 - q_2 + 1}{2}\right], \tag{26}$$

where $0 \leq \check{E}_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \leq 1$.

4. Fuzzy Weighted Averaging Aggregation Operators

In this section, we propose some different types of (q_1,q_2)-LDF weighted averaging aggregation operators. First, we define the (q_1,q_2)-linear Diophantine fuzzy weighted averaging aggregation operator.

Definition 13. Let $\Upsilon_1 = \{\langle \varsigma_1, \psi_1\rangle, \langle q_{11}, q_{21}\rangle\}$ and $\Upsilon_2 = \{\langle \varsigma_2, \psi_2\rangle, \langle q_{12}, q_{22}\rangle\}$ be two (q_1,q_2)-LDFNs and $\lambda > 0$,

$$\begin{aligned}\Upsilon_1 \oplus \Upsilon_2 &= (\langle \varsigma_1 + \varsigma_2 - \varsigma_1\varsigma_2, \psi_1\psi_2\rangle, \langle q_{11} + q_{12} - q_{11}q_{12}, q_{21}q_{22}\rangle),\\ \lambda\Upsilon_1 &= \left(\left\langle 1 - (1-\varsigma_1)^\lambda, {\psi_1}^\lambda\right\rangle, \left\langle 1 - (1-q_{11})^\lambda, {q_{21}}^\lambda\right\rangle\right).\end{aligned} \tag{27}$$

For the sake of simplicity, the set of (q_1,q_2)-LDFNs ((q_1,q_2)-*LDFNs*) on E is denoted by (q_1,q_2)-*LDFN*(E).

Example 3. Let $\Upsilon_1 = (\langle 1, 0.8\rangle, \langle 0.3, 0.5\rangle)$ and $\Upsilon_2 = (\langle 0.5, 0.9\rangle, \langle 0.5, 0.3\rangle)$ be two alternative with (q_1,q_2)-*LDFNs*.

$$\Upsilon_1 \oplus \Upsilon_2 = (\langle 0.5, 0.72\rangle, \langle 0.5, 0.15\rangle). \tag{28}$$

If $\lambda = 0.6$, then

$$\lambda\Upsilon_1 = (\langle 1, 0.875\rangle, \langle 0.193, 0.66\rangle). \tag{29}$$

Definition 14. The (q_1,q_2)-*LDFWAA* operator on "*n*" numbers of (q_1,q_2)-*LDFNs* on the set E is defined with the help of this transformation Ω: (q_1,q_2)-*LDFN*$(E) \longrightarrow (q_1,q_2)$-*LDFN*$(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^n \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1\rangle, \langle q_{11}, q_{21}\rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2\rangle, \langle q_{12}, q_{22}\rangle), \ldots\ldots\ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n\rangle, \langle q_{1n}, q_{2n}\rangle)\}$ are (q_1,q_2)-*LDFNs*,

$$(q_1,q_2)\text{-}LDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^n \omega_j\Upsilon_j. \tag{30}$$

Theorem 1. *The (q_1,q_2)-LDFWAA operator on "n" numbers of (q_1,q_2)-LDFNs on the set E is defined with the help of this transformation Ω: (q_1,q_2)-LDFN$(E) \longrightarrow (q_1,q_2)$-LDFN$(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ with $\sum_{j=1}^n \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1\rangle, \langle q_{11}, q_{21}\rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2\rangle, \langle q_{12}, q_{22}\rangle), \ldots\ldots\ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n\rangle, \langle q_{1n}, q_{2n}\rangle)\}$ are (q_1,q_2) are (q_1,q_2)-LDFNs,*

$$(q_1,q_2)\text{-}LDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^n \omega_j\Upsilon_j = \left(\left\langle 1 - \prod_{j=1}^n \left(1-\varsigma_j\right)^{\omega_j}, \prod_{j=1}^n {\psi_j}^{\omega_j}\right\rangle, \left\langle 1 - \prod_{j=1}^n \left(1-q_{1j}\right)^{\omega_j}, \prod_{j=1}^n {q_{2j}}^{\omega_j}\right\rangle\right). \tag{31}$$

This operator can easily be proved with support of (q_1,q_2)-*LDFNs* operations and mathematical-induction. Here, ς and ψ are representing the membership and nonmembership function. ω is called weight function, Υ_j are (q_1,q_2)-*LDFNs*, where $j \in N$.

Proof 2. As evident from Definition 1, (q_1,q_2)-*LDFWAA*$_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$ is a (q_1,q_2)-*LDFN*. By utilizing mathematical induction, it can be seen that the second part is also true. If $n = 2$, then we have

$$
\begin{aligned}
(q_1,q_2)\text{-}LDFWAA_\omega(\Upsilon_1,\Upsilon_2) &= \omega_1\Upsilon_1 \oplus \omega_2\Upsilon_2 \\
&= ((1-(1-\varsigma_1)^{\omega_1},\psi_1^{\omega_1}),(1-(1-q_{11})^{\omega_1},q_{21}^{\,\omega_1}))\oplus((1-(1-\varsigma_2)^{\omega_2},\psi_2^{\omega_2}),(1-(1-q_{12})^{\omega_2},q_{22}^{\,\omega_2})) \\
&= ((1-(1-\varsigma_1)^{\omega_1}+1-(1-\varsigma_2)^{\omega_2}-(1-(1-\varsigma_1)^{\omega_1})(1-(1-\varsigma_2)^{\omega_2}),\psi_1^{\omega_1}\psi_2^{\omega_2}),(1-(1-q_{11})^{\omega_1} \\
&\quad +1-(1-q_{12})^{\omega_2}-(1-(1-q_{11})^{\omega_1})(1-(1-q_{12})^{\omega_2}),q_{21}^{\,\omega_1}q_{22}^{\,\omega_2})) \\
&= \left(\left(1-\prod_j^2(1-\varsigma_j)^{\omega_j},\prod_j^2\psi_j^{\omega_j}\right),\left(1-\prod_j^2(1-q_{1j})^{\omega_j},\prod_j^2 q_{2j}^{\,\omega_j}\right)\right).
\end{aligned}
\tag{32}
$$

Suppose the above expression is true for $n=k$, that is,

$$
(q_1,q_2)\text{-}LDFWAA_\omega(\Upsilon_1,\ldots,\Upsilon_k) = \left(\left(1-\prod_j^k(1-\varsigma_j)^{\omega_j},\prod_j^k\psi_j^{\omega_j}\right),\left(1-\prod_j^k(1-q_{1j})^{\omega_j},\prod_j^k q_{2j}^{\,\omega_j}\right)\right). \tag{33}
$$

We need to prove true for $n=k+1$,

$$
\begin{aligned}
(q_1,q_2)\text{-}LDFWAA_\omega(\Upsilon_1,\ldots,\Upsilon_{k+1}) &= \left(\left(1-\prod_j^k(1-\varsigma_j)^{\omega_j},\prod_j^k\psi_j^{\omega_j}\right),\left(1-\prod_j^k(1-q_{1j})^{\omega_j},\prod_j^k q_{2j}^{\,\omega_j}\right)\right) \\
&\quad \oplus\left(\left(1-(1-\varsigma_{k+1})^{\omega_{k+1}},\psi_{k+1}^{\omega_{k+1}}\right),\left(1-(1-q_{1k+1})^{\omega_{k+1}},q_{2k+1}^{\,\omega_{k+1}}\right)\right) \\
&= \left(\left(1-\prod_j^{k+1}(1-\varsigma_j)^{\omega_j},\prod_j^{k+1}\psi_j^{\omega_j}\right),\left(1-\prod_j^{k+1}(1-q_{1j})^{\omega_j},\prod_j^{k+1} q_{2j}^{\,\omega_j}\right)\right).
\end{aligned}
\tag{34}
$$

This implies (30) is true.

We would now like to introduce the (q_1,q_2)-LDF ordered WAA (*LDFOWAA*) operator. □

Definition 15. The (q_1,q_2)-*LDFOWAA* operator on "*n*" numbers of (q_1,q_2)-*LDFNs* is defined with the help of this transformation Ω: (q_1,q_2)-*LDFN* $(E)\longrightarrow(q_1,q_2)$-*LDFN* (E) associated with $\omega=(\omega_1,\omega_2,\omega_3,\ldots,\omega_n)^T$ with $\sum_{j=1}^n\omega_j=1$ and it can be computed as follows: when $\{\Upsilon_1=(\langle\varsigma_1,\psi_1\rangle,\langle q_{11},q_{21}\rangle),\ \Upsilon_2=(\langle\varsigma_2,\psi_2\rangle,\langle q_{12},q_{22}\rangle),\ldots,\ \Upsilon_n=(\langle\varsigma_n,\psi_n\rangle,\langle q_{1n},q_{2n}\rangle)\}$ are (q_1,q_2)-*LDFN*,

$$
(q_1,q_2)\text{-}LDFOWAA_\omega(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots\ldots,\Upsilon_n)=\prod_{j=1}^n\omega_j\Upsilon_{\sigma(j)}, \tag{35}
$$

where $(\sigma(1),\sigma(2),\sigma(3),\ldots\ldots\ldots\ldots,\sigma(n))$ is the arrangement of $j\in N$, for which $\Upsilon_{\sigma(j-1)}\geq\Upsilon_{\sigma(j)}$, for all $j\in N$.

Theorem 2. *The* (q_1,q_2)-*LDFOWAA operator on "n" numbers of* (q_1,q_2)-*LDFNs is defined with the help of this transformation* Ω: (q_1,q_2)-*LDFN* $(E)\longrightarrow(q_1,q_2)$-*LDFN* (E) *associated with* $\omega=(\omega_1,\omega_2,\omega_3,\ldots,\omega_n)^T$ *with* $\sum_{j=1}^n\omega_j=1$ *and it can be computed as follows: when* $\{\Upsilon_1=(\langle\varsigma_1,\psi_1\rangle,\langle q_{11},q_{21}\rangle),\ \Upsilon_2=(\langle\varsigma_2,\psi_2\rangle,\langle q_{12},q_{22}\rangle),\ldots,\ \Upsilon_n=(\langle\varsigma_n,\psi_n\rangle,\langle q_{1n},q_{2n}\rangle)\}$ *are* (q_1,q_2)-*LDFNs,*

$$
\begin{aligned}
(q_1,q_2)\text{-}LDFOWAA_\omega(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots\ldots,\Upsilon_n) &= \prod_{j=1}^n\omega_j\Upsilon_{\sigma(j)} \\
&= \left(\left\langle 1-\prod_{j=1}^n(1-\varsigma_{\sigma(j)})^{\omega_j},\prod_{j=1}^n\psi_{\sigma(j)}^{\ \omega_j}\right\rangle,\left\langle 1-\prod_{j=1}^n(1-q_{1\sigma(j)})^{\omega_j},\prod_{j=1}^n q_{2\sigma(j)}^{\ \omega_j}\right\rangle\right),
\end{aligned}
\tag{36}
$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots\ldots\ldots\ldots, \sigma(n))$ *is the arrangement of* $j \in N$*, for which* $\Upsilon_{\sigma(j-1)} \geq \Upsilon_{\sigma(j)}$*, for all* $j \in N$.

Proof 3. This can also be calculated utilizing the method based on the normal distribution [30].

The (q_1, q_2)-linear Diophantine fuzzy hybrid weighted averaging aggregation (q_1, q_2)-LDFHWAA) operator is now ready for introduction. □

Definition 16. The (q_1, q_2)-*LDFHWAA* operator on "n" numbers of (q_1, q_2)-*LDFNs* is defined with the help of this transformation
Ω: (q_1, q_2)-*LDFN*$(E) \longrightarrow (q_1, q_2)$-*LDFN*$(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle), \ldots\ldots\ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n \rangle, \langle q_{1n}, q_{2n} \rangle)\}$ are (q_1, q_2)-*LDFNs*,

$$(q_1, q_2)\text{-}LDFHWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \omega_j \Upsilon^\star_{\sigma(j)}, \tag{37}$$

where $\Upsilon^\star_{\sigma(j)}$ is biggest j th weighted (q_1, q_2)-linear Diophantine fuzzy values $\Upsilon^\star_j$ $(\Upsilon^\star_j = (\Upsilon_j)^{n\omega_j}, j \in N)$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ be the weights of $\Upsilon^\star_j$ by means of $\omega > 0$ with $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 3. *The* (q_1, q_2)-*LDFHWAA operator on "n" numbers of* (q_1, q_2)-*LDFNs is defined with the help of this transformation*
Ω: (q_1, q_2)-*LDFN*$(E) \longrightarrow (q_1, q_2)$-*LDFN*$(E)$ *associated with* $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ *with* $\sum_{j=1}^{n} \omega_j = 1$ *and it can be computed as follows: when* $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle), \ldots\ldots\ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n \rangle, \langle q_{1n}, q_{2n} \rangle)\}$ *are* (q_1, q_2)-*LDFNs,*

$$\begin{aligned}(q_1, q_2)\text{-}LDFHWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) &= \prod_{j=1}^{n} \omega_j \Upsilon^\star_{\sigma(j)} \\ &= \left(\left\langle 1 - \prod_{j=1}^{n} \left(1 - \varsigma^\star_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \psi^\star_{\sigma(j)} \omega_j \right\rangle, \left\langle 1 - \prod_{j=1}^{n} \left(1 - q_1^\star \sigma(j)\right)^{\omega_j}, \prod_{j=1}^{n} q_2^\star \sigma(j)^{\omega_j} \right\rangle\right),\end{aligned} \tag{38}$$

where $\Upsilon^\star_{\sigma(j)}$ *is biggest j th weighted* (q_1, q_2)-*LDF values* $\Upsilon^\star_j$ $(\Upsilon^\star_j = n\omega_j \Upsilon_j, j \in N)$ *and* $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ *be the weights of* $\Upsilon^\star_j$ *by means of* $\omega > 0$ *with* $\sum_{j=1}^{n} \omega_j = 1$.

Proof 4. The proof follows similar steps as those used in the operations of *IFSs*, and as such, is excluded.

It is interesting to note that if $\omega = (1/n, (1/n), (1/n), \ldots\ldots\ldots, 1/n)$, then (q_1, q_2)-*LDFWAA* and (q_1, q_2)-*LDFOWAA* operators are considered to be exceptional cases of (q_1, q_2)-*LDFHWAA* operator. So it concludes that (q_1, q_2)-*LDFHWAA* operators are the extension of (q_1, q_2)-*LDFWAA* and (q_1, q_2)-*LDFOWAA* operators. □

5. FWGA Operators

In this section, we propose some different types of (q_1, q_2)-LDFWGA operators. First, we define the (q_1, q_2)-LDFWGA operator.

Definition 17. Let $\Upsilon_1 = \{\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle\}$ and $\Upsilon_2 = \{\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle\}$ be two (q_1, q_2)-LDF numbers and $\lambda > 0$

$$\begin{aligned}\Upsilon_1 \otimes \Upsilon_2 &= (\langle \varsigma_1 \varsigma_2, \psi_1 + \psi_2 - \psi_1 \psi_2 \rangle, \langle q_{11} q_{12}, q_{21} + q_{22} - q_{21} q_{22} \rangle), \\ \Upsilon_1{}^\lambda &= \left(\left\langle \varsigma_1{}^\lambda, 1 - (1 - \psi_1)^\lambda \right\rangle, \left\langle q_{11}{}^\lambda, 1 - (1 - q_{21})^\lambda \right\rangle\right).\end{aligned} \tag{39}$$

Example 4. Let $\Upsilon_1 = (\langle 0.8, 1 \rangle, \langle 0.5, 0.3 \rangle)$ and $\Upsilon_2 = (\langle 0.9, 0.5 \rangle, \langle 0.3, 0.5 \rangle)$ be two alternative with (q_1, q_2)-*LDFNs*.

$$\Upsilon_1 \otimes \Upsilon_2 = (\langle 0.72, 0.5 \rangle, \langle 0.15, 0.5 \rangle). \tag{40}$$

If $\lambda = 0.6$, then

$$\Upsilon_1{}^\lambda = (\langle 0.875, 1 \rangle, \langle 0.66, 0.193 \rangle). \tag{41}$$

Definition 18. The (q_1, q_2)-*LDFWGA* operator on "n" numbers of (q_1, q_2)-*LDFNs* on the set E is defined with the help of this transformation Ω: (q_1, q_2)-*LDFN*$(E) \longrightarrow (q_1, q_2)$-*LDFN*$(E)$ associated with weight vector

$\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ with $\sum_{j=1}^{n}\omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1\rangle, \langle q_{11}, q_{21}\rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2\rangle, \langle q_{12}, q_{22}\rangle), \ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n\rangle, \langle q_{1n}, q_{2n}\rangle)\}$ are (q_1, q_2)-*LDFNs*,

$$(q_1, q_2)\text{-}LDFWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon_j^{\,\omega_j}. \tag{42}$$

Theorem 4. *The* (q_1, q_2)*-LDFWGA operator on "n" numbers of* (q_1, q_2)*-LDFNs on the set E is defined with the help of this transformation* Ω: (q_1, q_2)*-LDFN* $(E) \longrightarrow (q_1, q_2)$*-LDFN* (E) *associated with weight vector* $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ *with* $\sum_{j=1}^{n}\omega_j = 1$ *and it can be computed as follows: when* $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1\rangle, \langle q_{11}, q_{21}\rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2\rangle, \langle q_{12}, q_{22}\rangle), \ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n\rangle, \langle q_{1n}, q_{2n}\rangle)\}$ *are* (q_1, q_2)*-LDFNs,*

$$(q_1, q_2)\text{-}LDFWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon_j^{\,\omega_j} = \left(\left\langle \prod_{j=1}^{n} \varsigma_j^{\,\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \psi_j\right)^{\omega_j} \right\rangle, \left\langle \prod_{j=1}^{n} q_{1j}^{\,\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - q_{2j}\right)^{\omega_j} \right\rangle\right). \tag{43}$$

This operator can easily be proved with support of (q_1, q_2)-*LDFNs* operations and mathematical induction. Here, ς and ψ are representing the membership and nonmembership function. ω is called weight function, Υ_j are (q_1, q_2)-*LDFNs*, where $j \in N$.

Proof 5. As evident from Definition 18, (q_1, q_2)-$LDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$ is a (q_1, q_2)-*LDFN*. By utilizing mathematical induction, it can be seen that the second part is also true. If $n = 2$, then we have

$$\begin{aligned}
(q_1, q_2)\text{-}LDFWAA_\omega(\Upsilon_1, \Upsilon_2) &= \omega_1 \Upsilon_1 \oplus \omega_2 \Upsilon_2 \\
&= \left(\left(\varsigma_1^{\omega_1}, 1 - (1 - \psi_1)^{\omega_1}\right), \left(q_{11}^{\,\omega_1}, 1 - (1 - q_{21})^{\omega_1}\right)\right) \oplus \left(\left(\varsigma_2^{\omega_2}, 1 - (1 - \psi_2)^{\omega_2}\right), \left(q_{12}^{\,\omega_2}, 1 - (1 - q_{22})^{\omega_2}\right)\right) \\
&= \left(\left(\varsigma_1^{\omega_1}\varsigma_2^{\omega_2}, 1 - (1 - \psi_1)^{\omega_1} + 1 - (1 - \psi_2)^{\omega_2} - \left(1 - (1 - \psi_1)^{\omega_1}\right)\left(1 - (1 - \psi_2)^{\omega_2}\right)\right),\right. \\
&\quad \left.\left(q_{11}^{\,\omega_1} q_{12}^{\,\omega_2}, 1 - (1 - q_{21})^{\omega_1} + 1 - (1 - q_{22})^{\omega_2} - \left(1 - (1 - q_{21})^{\omega_1}\right)\left(1 - (1 - q_{22})^{\omega_2}\right)\right)\right) \\
&= \left(\left(\prod_{j}^{2} \varsigma_j^{\omega_j}, 1 - \prod_{j}^{2}\left(1 - \psi_j\right)^{\omega_j}\right), \left(\prod_{j}^{2} q_{1j}^{\,\omega_j}, 1 - \prod_{j}^{2}\left(1 - q_{2j}\right)^{\omega_j}\right)\right).
\end{aligned} \tag{44}$$

Suppose the above expression is true for $n = k$, that is,

$$(q_1, q_2)\text{-}LDFWAA_\omega(\Upsilon_1, \ldots, \Upsilon_k) = \left(\left(\prod_{j}^{k} \varsigma_j^{\omega_j}, 1 - \prod_{j}^{k}\left(1 - \psi_j\right)^{\omega_j}\right), \left(\prod_{j}^{k} q_{1j}^{\,\omega_j}, 1 - \prod_{j}^{k}\left(1 - q_{2j}\right)^{\omega_j}\right)\right). \tag{45}$$

We need to prove true for $n = k + 1$,

$$\begin{aligned}
(q_1, q_2)\text{-}LDFWAA_\omega(\Upsilon_1, \ldots, \Upsilon_{k+1}) &= \left(\left(\prod_{j}^{k} \varsigma_j^{\omega_j}, 1 - \prod_{j}^{k}\left(1 - \psi_j\right)^{\omega_j}\right), \left(\prod_{j}^{k} q_{1j}^{\,\omega_j}, 1 - \prod_{j}^{k}\left(1 - q_{2j}\right)^{\omega_j}\right)\right) \\
&\quad \oplus \left(\left(\varsigma_{k+1}^{\omega_{k+1}}, 1 - (1 - \psi_{k+1})^{\omega_{k+1}}\right), \left(q_{1k+1}^{\,\omega_{k+1}}, 1 - (1 - q_{2k+1})^{\omega_{k+1}}\right)\right) \\
&= \left(\left(\prod_{j}^{k+1} \varsigma_j^{\omega_j}, 1 - \prod_{j}^{k+1}\left(1 - \psi_j\right)^{\omega_j}\right), \left(\prod_{j}^{k+1} q_{1j}^{\,\omega_j}, 1 - \prod_{j}^{k+1}\left(1 - q_{2j}\right)^{\omega_j}\right)\right).
\end{aligned} \tag{46}$$

This implies (43) is true. □

Definition 19. The (q_1, q_2)-LDF ordered WGA ((q_1, q_2)-*LDFOWGA*) operator on $''n''$ numbers of (q_1, q_2)-*LDFNs* is defined with the help of this transformation Ω: (q_1, q_2)-*LDFN* $(E) \longrightarrow (q_1, q_2)$-*LDFN* (E) associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle), \ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n \rangle, \langle q_{1n}, q_{2n} \rangle)\}$ are (q_1, q_2)-*LDFN*,

$$(q_1, q_2)\text{-}LDFOWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon_{\sigma(j)}{}^{\omega_j}, \tag{47}$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots\ldots\ldots, \sigma(n))$ is the arrangement of $j \in N$, for which $\Upsilon_{\sigma(j-1)} \geq \Upsilon_{\sigma(j)}$, for all $j \in N$.

Theorem 5. *The (q_1, q_2)-LDFOWGA operator on $''n''$ numbers of (q_1, q_2)-LDFNs is defined with the help of this transformation Ω: (q_1, q_2)-LDFN $(E) \longrightarrow (q_1, q_2)$-LDFN (E) associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle), \ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n \rangle, \langle q_{1n}, q_{2n} \rangle)\}$ are (q_1, q_2)-LDFN,*

$$\begin{aligned}(q_1, q_2)\text{-}LDFOWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) &= \prod_{j=1}^{n} \Upsilon_{\sigma(j)}{}^{\omega_j} \\ &= \left(\left\langle \prod_{j=1}^{n} \varsigma_{\sigma(j)}{}^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \psi_{\sigma(j)}\right)^{\omega_j} \right\rangle, \left\langle \prod_{j=1}^{n} q_{1\sigma(j)}{}^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - q_{2\sigma(j)}\right)^{\omega_j} \right\rangle \right),\end{aligned} \tag{48}$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots\ldots\ldots, \sigma(n))$ is the arrangement of $j \in N$, for which $\Upsilon_{\sigma(j-1)} \geq \Upsilon_{\sigma(j)}$, for all $j \in N$.

Proof 6. This can also be calculated utilizing the method based on the normal distribution [30]. □

Definition 20. The (q_1, q_2)-*LDFHWGA* operator on $''n''$ numbers of (q_1, q_2)-*LDFNs* is defined with the help of this transformation Ω: (q_1, q_2)-*LDFN* $(E) \longrightarrow (q_1, q_2)$-*LDFN* (E) associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle), \ldots\ldots\ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n \rangle, \langle q_{1n}, q_{2n} \rangle)\}$ are (q_1, q_2)-*LDFNs*,

$$(q_1, q_2)\text{-}LDFHWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon^{\star}_{\sigma(j)} \omega_j, \tag{49}$$

where $\Upsilon^{\star}_{\sigma(j)}$ is biggest j th weighted (q_1, q_2)-LDF values $\Upsilon^{\star}_j$ $(\Upsilon^{\star}_j = (\Upsilon_j)^{n\omega_j}, j \in N)$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ be the weights of $\Upsilon^{\star}_j$ by means of $\omega > 0$ with $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 6. *The (q_1, q_2)-LDFHWGA operator on $''n''$ numbers of (q_1, q_2)-LDFNs is defined with the help of this transformation Ω: (q_1, q_2)-LDFN $(E) \longrightarrow (q_1, q_2)$-LDFN (E) associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: when $\{\Upsilon_1 = (\langle \varsigma_1, \psi_1 \rangle, \langle q_{11}, q_{21} \rangle), \Upsilon_2 = (\langle \varsigma_2, \psi_2 \rangle, \langle q_{12}, q_{22} \rangle), \ldots, \Upsilon_n = (\langle \varsigma_n, \psi_n \rangle, \langle q_{1n}, q_{2n} \rangle)\}$ are (q_1, q_2)-LDFNs,*

$$\begin{aligned}(q_1, q_2)\text{-}LDFHWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) &= \prod_{j=1}^{n} \Upsilon^{\star}_{\sigma(j)} \omega_j \\ &= \left(\left\langle \prod_{j=1}^{n} \varsigma^{\star}_{\sigma(j)} \omega_j, 1 - \prod_{j=1}^{n} \left(1 - \psi^{\star}_{\sigma(j)}\right)^{\omega_j} \right\rangle, \left\langle \prod_{j=1}^{n} q_{1\sigma(j)}^{\star} \omega_j, 1 - \prod_{j=1}^{n} \left(1 - q_{2\sigma(j)}^{\star}\right)^{\omega_j} \right\rangle \right),\end{aligned} \tag{50}$$

where $\Upsilon^{\star}_{\sigma(j)}$ is biggest j th weighted (q_1, q_2)-LDF values $\Upsilon^{\star}_j$ $(\Upsilon^{\star}_j = (\Upsilon_j)^{n\omega_j}, j \in N)$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ be the weights of $\Upsilon^{\star}_j$ by means of $\omega > 0$ with $\sum_{j=1}^{n} \omega_j = 1$.

It is interesting to note that if $\omega = (1/n, (1/n), (1/n), \ldots, 1/n)$, then (q_1, q_2)-*LDFWGA* and (q_1, q_2)-*LDFOWGA* operators are considered to be exceptional cases of

(q_1, q_2)-*LDFHWGA* operator. So it concludes that (q_1, q_2)-*LDFHWGA* operators are the extension of (q_1, q_2)-*LDFWGA* and (q_1, q_2)-*LDFOWGA* operators.

6. MADM Approach Using Suggested Techniques

The *MADM* technique is highly effective and well-suited for selecting the optimal choice from a limited set of possibilities due to its structure. To enhance the effectiveness and quality of previously proposed methods, we introduce a section on the *MADM* technique procedure incorporating four appropriate operators: the (q_1, q_2)-*LDFWAA* operator, (q_1, q_2)-*LDFOWAA* operator, and (q_1, q_2)-*LDFHWAA* operator. To assess some real-world issues, our objective is to calculate the decision-making process.

As a collection of finite values of alternatives, we take into consideration $\mathfrak{R} = \{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_m\}$. In addition, we choose a finite set of attributes, including, $\hat{U} = \{\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_n\}$ are chosen along with a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j > 0$ with $\sum_{j=1}^{n} \omega_j = 1$, for every alternative. Additionally, in order to calculate the matrix that assesses the optimal choice after taking the decision-making process into account, we hope to assign the (q_1, q_2)-*LDF* values to each alternative, observed that ς_j and ψ_j denote the positive and negative grades, where α_j and β_j are reference parameters corresponding to alternative $(\mathfrak{R}_j)$ that satisfy the attribute $(\hat{U}_j)$ provided by the decision makers, where $0 \le q_{1j}\varsigma_j(x) + q_{2j}\psi_j(x) \le 1$ and $0 \le q_{1j} + q_{2j} \le 1$. Additionally, we stated the refusal degree $q_{3j}\pi_j(x) = 1 - q_{1j}\varsigma_j(x) - q_{2j}\psi_j(x)$. As a result, in order to accomplish the aforementioned approach, we take into account a few real-world applications and attempt to assess them using theoretical frameworks.

6.1. The Suggested Algorithm.

The primary impact of this subsection is to assess a process for illustrating the problem that will be addressed in the following section. The primary steps of the decision-making approach are outlined below:

- Step 1. Determine a team matrix by incorporating their values into the (q_1, q_2)-*LDFN* form. Additionally, while we assign the values, we have two opinions "profit and cost," such as if we have cost-type data, then our first priority is to normalize it otherwise not.
- Step 2. Using the six various types of operators "(q_1, q_2)-*LDFWAA* operator, (q_1, q_2)-*LDFOWAA* operator, (q_1, q_2)-*LDFHWAA* operator, (q_1, q_2)-*LDFWGA* operator, (q_1, q_2)-*LDFOWGA* operator, and (q_1, q_2)-*LDFHWGA* operator" aggregate the collection of data into a singleton set such that

$$\begin{aligned}
&(q_1, q_2)\text{-}LDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \omega_j \Upsilon_j \\
&= \left(\left\langle 1 - \prod_{j=1}^{n} \left(1 - \varsigma_j\right)^{\omega_j}, \prod_{j=1}^{n} \psi_j{}^{\omega_j} \right\rangle, \left\langle 1 - \prod_{j=1}^{n} \left(1 - q_{1j}\right)^{\omega_j}, \prod_{j=1}^{n} q_{2j}{}^{\omega_j} \right\rangle \right), \\
&(q_1, q_2)\text{-}LDFOWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \omega_j \Upsilon_{\sigma(j)} \\
&= \left(\left\langle 1 - \prod_{j=1}^{n} \left(1 - \varsigma_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \psi_{\sigma(j)}{}^{\omega_j} \right\rangle, \left\langle 1 - \prod_{j=1}^{n} \left(1 - q_{1\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} q_{2\sigma(j)}{}^{\omega_j} \right\rangle \right), \\
&(q_1, q_2)\text{-}LDFHWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \omega_j \Upsilon^\star_{\sigma(j)} \\
&= \left(\left\langle 1 - \prod_{j=1}^{n} \left(1 - \varsigma^\star_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \psi^\star_{\sigma(j)}\omega_j \right\rangle, \left\langle 1 - \prod_{j=1}^{n} \left(1 - q^\star_{1\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} q^\star_{2\sigma(j)}\omega_j \right\rangle \right), \\
&(q_1, q_2)\text{-}LDFWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon_j{}^{\omega_j} \\
&= \left(\left\langle \prod_{j=1}^{n} \varsigma_j{}^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \psi_j\right)^{\omega_j} \right\rangle, \left\langle \prod_{j=1}^{n} q_{1j}{}^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - q_{2j}\right)^{\omega_j} \right\rangle \right), \\
&(q_1, q_2)\text{-}LDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon_{\sigma(j)}{}^{\omega_j} \\
&= \left(\left\langle \prod_{j=1}^{n} \varsigma_{\sigma(j)}{}^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \psi_{\sigma(j)}\right)^{\omega_j} \right\rangle, \left\langle \prod_{j=1}^{n} q_{1\sigma(j)}{}^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - q_{2\sigma(j)}\right)^{\omega_j} \right\rangle \right). \\
&(q_1, q_2)\text{-}LDFHWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots\ldots\ldots, \Upsilon_n) = \prod_{j=1}^{n} \Upsilon^\star_{\sigma(j)}\omega_j \\
&= \left(\left\langle \prod_{j=1}^{n} \varsigma^\star_{\sigma(j)}\omega_j, 1 - \prod_{j=1}^{n} \left(1 - \psi^\star_{\sigma(j)}\right)^{\omega_j} \right\rangle, \left\langle \prod_{j=1}^{n} q^\star_{1\sigma(j)}\omega_j, 1 - \prod_{j=1}^{n} \left(1 - q^\star_{2\sigma(j)}\right)^{\omega_j} \right\rangle \right).
\end{aligned} \tag{51}$$

- Step 3. Determine the aggregated theories with respect to different score values, such as

$$\begin{aligned} S_{(q_1,q_2)\text{-}LDFN}(\Upsilon) &= \frac{1}{2}\left[Ϛ_{\Upsilon} - \psi_{\Upsilon} + q_1 - q_2\right], \\ \$_{(q_1,q_2)\text{-}LDFN}(\Upsilon) &= \frac{1}{2}\left[{Ϛ_{\Upsilon}}^2 - {\psi_{\Upsilon}}^2 + {q_1}^2 - {q_2}^2\right], \\ \check{E}_{(q_1,q_2)\text{-}LDFN}(\Upsilon) &= \frac{1}{2}\left[\frac{Ϛ_{\Upsilon} - \psi_{\Upsilon} + 1}{2} + \frac{q_1 - q_2 + 1}{2}\right]. \end{aligned} \tag{52}$$

where $-1 \le S_{(q_1,q_2)\text{-}LDFN}(\Upsilon),\ \$_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \le 1,\ 0 \le \check{E}_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \le 1$.

In the event that the score function is not successful, then the accuracy function will be used like

$$\begin{aligned} H_{(q_1,q_2)\text{-}LDFN}(\Upsilon) &= \frac{1}{2}\left[\frac{Ϛ_{\Upsilon} + \psi_{\Upsilon}}{2} + q_1 + q_2\right], \\ И_{(q_1,q_2)\text{-}LDFN}(\Upsilon) &= \frac{1}{2}\left[\frac{{Ϛ_{\Upsilon}}^2 + {\psi_{\Upsilon}}^2}{2} + {q_1}^2 + {q_2}^2\right], \end{aligned} \tag{53}$$

where $0 \le H_{(q_1,q_2)\text{-}LDFN}(\Upsilon), И_{(q_1,q_2)\text{-}LDFN}(\Upsilon) \le 1$.

- Step 4. Try to identify the standout among the alternatives by analyzing the ranking values based on the score values.

To improve the value of the assessed techniques and enable the practical application of the aforementioned procedure, we take into consideration a number of numerical examples that demonstrate the superiority and validity of the invented operators. The suggested algorithm's geometrical interpretation is presented in the form of Figure 3.

6.2. Numerical Example. In this section, we examine the blood pressure levels, also known as the assessment of the patient's high blood pressure conditions. This method is utilized to determine the optimal approach for import or export while considering environmental impact. In this example, we consider four different types of green supply chains and evaluate the best one based on the proposed theory, for instance.

- $\mathfrak{R}_1$: The conditions associated with very high risk are more likely to occur at severely elevated blood pressure levels (e.g., 180/120 mmHg or higher). At these levels, the risk of catastrophic health events like heart attack or stroke increases dramatically, necessitating urgent medical intervention.
- $\mathfrak{R}_2$: The conditions categorized as high risk might become more probable at moderately elevated blood pressure levels (e.g., 140/90 mmHg or slightly higher). They signify significant concern and need for intervention but are often manageable with appropriate treatment.
- $\mathfrak{R}_3$: Risk of Illness indicates that a person is at some level of risk for developing illnesses or complications related to hypertension. The risk can vary from low to high, depending on factors such as blood pressure levels, lifestyle, genetics, and other health conditions. Individuals in this category already have hypertension or are on the verge of developing it. Their risk of hypertension-related illnesses like heart disease, stroke, kidney disease, and others is elevated. This group may require lifestyle modifications and possibly medication to manage or reduce their risk e.g. A person with blood pressure readings consistently around 140/90 mmHg is at risk of developing complications from hypertension if their condition is not managed.
- $\mathfrak{R}_3$: Not Susceptible refers to individuals who are less likely to develop hypertension or its associated illnesses. This could be due to genetic factors, lifestyle, or other protective factors that reduce their likelihood of developing high blood pressure. People in this category typically have normal blood pressure and maintain a healthy lifestyle that reduces their risk of developing hypertension. They might have a family history free of hypertension or other protective factors like a balanced diet, regular exercise, and low-stress levels e.g. A person with a blood pressure reading consistently around 110/70 mmHg, who exercises regularly and follows a healthy diet, might be considered not susceptible to hypertension at the current stage.
- $\mathfrak{R}_4$: Normal refers to having a blood pressure level that is within the standard healthy range, typically defined as a systolic pressure (the top number) less than 120 mmHg and a diastolic pressure (the bottom number) less than 80 mmHg. Individuals with normal blood pressure are considered to be at a low risk for developing hypertension-related illnesses. Maintaining this normal range involves a healthy lifestyle, which includes a balanced diet, regular physical activity, avoiding excessive salt and alcohol, and managing stress e.g. A person with a blood pressure of 115/75 mmHg falls within the normal range. They are not currently at risk for hypertension-related illnesses, provided they maintain a healthy lifestyle.

6.3. Summary of Differences. The "very high and high-risk illnesses" related to hypertension are serious, the key differences lie in the severity of the conditions, the immediacy with which they impact health, and the urgency of medical intervention required. High-risk conditions are generally more gradual and manageable, whereas very high-risk conditions pose an immediate threat to life and health, often requiring urgent care. The "risk of illness" indicates some degree of vulnerability to hypertension-related complications, "not susceptible" suggests a lower chance of developing hypertension, and "normal" refers to a healthy blood pressure level with no immediate risk of hypertension.

Hypertension, or high blood pressure, is a common health issue that can affect anyone, but certain groups are more vulnerable due to various factors. Understanding these factors can help identify those at greater risk and guide preventative measures. Here are some key aspects:

We consider the four alternatives above, and to select the best one, we use the following attributes/criteria.

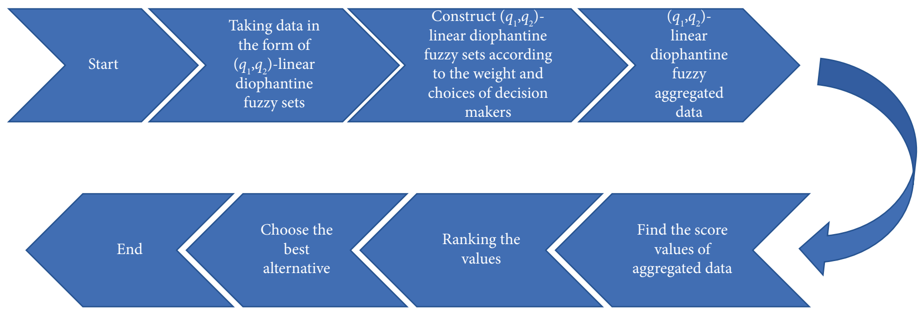

Figure 3: Geometrical interpretation of the proposed algorithm.

- $\hat{\mathrm{U}}_1$: Age: Blood pressure tends to increase with age due to the stiffening of arteries and blood vessels, making older adults more susceptible to hypertension.
- $\hat{\mathrm{U}}_2$: Genetics or Family History: If hypertension runs in the family, there's a higher chance that individuals will develop it due to genetic predisposition.
- $\hat{\mathrm{U}}_3$: Weight: Carrying extra weight increases the demand on the heart to pump blood, which can lead to higher blood pressure.
- $\hat{\mathrm{U}}_4$: Dietary Habits: Diets high in sodium, often from processed and fast foods, contribute significantly to the development of hypertension. Moreover, potassium helps balance sodium levels in the body, and low intake can increase the risk of hypertension.
- $\hat{\mathrm{U}}_5$: Stress: Prolonged stress can lead to temporary increases in blood pressure, and if unmanaged, this can contribute to chronic hypertension.

Under reference parameter mappings q_1 and q_2, α = very risky for illness and β = not very risky for illness are reference parameters corresponding to alternative $(\mathfrak{R}_j)$ that satisfy the attribute $(\hat{\mathrm{U}}_j)$ provided by the decision makers.

- Step 1. Determine a team matrix by incorporating their values into the (q_1,q_2)-*LDFN* form, see Table 3. Additionally, while we assign the values, we have two opinions "same type of data and different type of data", such as if we have different type of data, then our first priority is to normalize such that

$$\mathscr{L}_j = \begin{cases} \left(\left\langle \varsigma_j, \psi_j \right\rangle, \left\langle q_{1j}, q_{2j} \right\rangle\right), & \text{same type input data}, \\ \left(\left\langle \psi_j, \varsigma_j \right\rangle, \left\langle q_{2j}, q_{1j} \right\rangle\right), & \text{different type input data}. \end{cases} \tag{54}$$

  In this case, since the input data for all attributes is identical, there is no need to normalize the data. All alternatives and criteria in our specific problem are of the same nature with weight vector $(0.31, 17, 0.27, 0.24, 0.1)^T$.

- Step 2. Using the six various types of operators "(q_1,q_2)-*LDFWAA* operator, (q_1,q_2)-*LDFOWAA* operator, (q_1,q_2)-*LDFHWAA* operator, (q_1,q_2)-*LDFWGA* operator, (q_1,q_2)-*LDFOWGA* operator, and (q_1,q_2)-*LDFHWGA* operator" aggregate the collection of data into a singleton set, see Tables 4, 5, 6, 7, 8, 9.
- Step 3. Refer to Tables 10 and 11 to find the aggregated theory's score values such that:
- Step 4. Analyze the ranking values based on the score values and look for the standout alternative among the four; refer to Tables 12 and 13 with Figures 4, 5, 6.

Graphical representation of Tables 12 and 13.

By taking into account the theories of the (q_1,q_2)-*LDFWAA* operator, (q_1,q_2)-*LDFOWAA* operator, (q_1,q_2)-*LDFHWAA* operator, (q_1,q_2)-*LDFWGA* operator, (q_1,q_2)-*LDFOWGA* operator, and (q_1,q_2)-*LDFHWGA* operator we found that the most desirable decision is $\mathfrak{R}_4$. Note that, each operator receives the same rating results, these operators are also steady.

Table 14 lists the following benefits that the suggested operators have over the present operators.

- From these six aggregation operators, we found that the most desirable decision is $\mathfrak{R}_4$ and, $\mathfrak{R}_3$ and $\mathfrak{R}_1$ are the second best choices. Note that, each operator receives the same rating results, these operators are also steady. From Table 14, it can be easily seen that our proposed method is more flexible than other existing methods due to the reference parameter mappings.

- Given its greater versatility, expertise, and generality, the (q_1,q_2)-*LDF MADM* model can handle a greater number of decision-making problems with varying values while still meeting the requirements of *MADM* problems.

- Information in (q_1,q_2)-*LDFWAA* operators and ((q_1,q_2)-*LDFWGA*-operators is represented using (q_1,q_2)-*LDFNs*. Combining *FNs* with the pair of reference parameters mappings "(q_1,q_2)" yields the (q_1,q_2)-*LDFNs*, which provides all evaluation information. The suggested operator is more universal because the (q_1,q_2)-*LDFSs* manages both quantitative and qualitative data.

- It can be easily noticed from Table 15, when data are in the (q_1,q_2)-LDF information form, the existing classical operators presented are unable to address the problems. The suggested approach yields more precise and accurate results.

Table 3: Decision matrix of (q_1, q_2)-LDF information.

	$\hat{U}_1$	$\hat{U}_2$	$\hat{U}_3$	$\hat{U}_4$	$\hat{U}_5$
$\mathfrak{R}_1$	$\left(\begin{array}{c}\langle 0.9, 0.959\rangle,\\ \langle 0.841, 0.15\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.795, 0.825\rangle,\\ \langle 0.76, 0.23\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.9, 0.82\rangle,\\ \langle 0.5, 0.5\rangle)\end{array}\right)$	$\left(\begin{array}{c}\langle 0.9, 0.849\rangle,\\ \langle 0.536, 0.391\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.752, 0.8\rangle,\\ \langle 0.6, 0.337\rangle)\end{array}\right)$
$\mathfrak{R}_2$	$\left(\begin{array}{c}\langle 1, 0.731\rangle,\\ \langle 0.45, 0.53\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.9, 0.797\rangle,\\ \langle 0.6, 0.3\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 1, 0.98\rangle,\\ \langle 0.4, 0.4\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.9, 0.852\rangle,\\ \langle 0.447, 0.358\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 1, 0.8\rangle,\\ \langle 0.4, 0.6\rangle\end{array}\right)$
$\mathfrak{R}_3$	$\left(\begin{array}{c}\langle 1, 0.909\rangle,\\ \langle 0.3, 0.4\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.9, 0.881\rangle,\\ \langle 0.7, 0.26\rangle\end{array}\right)$	$\left(\begin{array}{c}(\langle 0.825, 0.758\rangle,\\ \langle 0.547, 0.37\rangle)\end{array}\right)$	$\left(\begin{array}{c}\langle 0.793, 0.887\rangle,\\ \langle 0.645, 0.263\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.78, 0.86\rangle,\\ \langle 0.7, 0.2\rangle\end{array}\right)$
$\mathfrak{R}_4$	$\left(\begin{array}{c}\langle 0.9, 0.866\rangle,\\ \langle 0.603, 0.334\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 0.9, 0.834\rangle,\\ \langle 0.225, 0.506\rangle\end{array}\right)$	$\left(\begin{array}{c}(\langle 0.9, 0.816\rangle,\\ \langle 0.9, 0.1\rangle)\end{array}\right)$	$\left(\begin{array}{c}\langle 1, 0.773\rangle,\\ \langle 0.342, 0.282\rangle\end{array}\right)$	$\left(\begin{array}{c}\langle 1, 0.884\rangle,\\ \langle 0.64, 0.306\rangle\end{array}\right)$

Table 4: (q_1, q_2)-*LDFWAA*.

$\mathfrak{R}_1$	$(\langle 0.8763, 0.8546\rangle, \langle 0.6705, 0.3047\rangle)$
$\mathfrak{R}_2$	$(\langle 1, 0.8405\rangle, \langle 0.4612, 0.4109\rangle)$
$\mathfrak{R}_3$	$(\langle 1, 0.8511\rangle, \langle 0.5793, 0.3071\rangle)$
$\mathfrak{R}_4$	$(\langle 1, 0.8257\rangle, \langle 0.6573, 0.2464\rangle)$

Table 5: (q_1, q_2)-*LDFOWAA*.

$\mathfrak{R}_1$	$(\langle 0.8556, 0.8510\rangle, \langle 0.6828, 0.2849\rangle)$
$\mathfrak{R}_2$	$(\langle 1, 0.8406\rangle, \langle 0.4710, 0.3981\rangle)$
$\mathfrak{R}_3$	$(\langle 1, 0.8484\rangle, \langle 0.6228, 0.2834\rangle)$
$\mathfrak{R}_4$	$(\langle 1, 0.8315\rangle, \langle 0.6520, 0.2526\rangle)$

Table 6: (q_1, q_2)-*LDFHWAA*.

$\mathfrak{R}_1$	$(\langle 0.8313, 0.8738\rangle, \langle 0.6139, 0.3681\rangle)$
$\mathfrak{R}_2$	$(\langle 1, 0.8758\rangle, \langle 0.4134, 0.4511\rangle)$
$\mathfrak{R}_3$	$(\langle 1, 0.8739\rangle, \langle 0.5043, 0.3831\rangle)$
$\mathfrak{R}_4$	$(\langle 1, 0.8618\rangle, \langle 0.6133, 0.3091\rangle)$

Table 7: (q_1, q_2)-*LDFWGA*.

$\mathfrak{R}_1$	$(\langle 0.8655, 0.8747\rangle, \langle 0.6234, 0.3660\rangle)$
$\mathfrak{R}_2$	$(\langle 0.9577, 0.8931\rangle, \langle 0.4517, 0.4382\rangle)$
$\mathfrak{R}_3$	$(\langle 0.8604, 0.8636\rangle, \langle 0.532959, 0.3245\rangle)$
$\mathfrak{R}_4$	$(\langle 0.9328, 0.8307\rangle, \langle 0.4989, 0.3209852\rangle)$

Table 8: (q_1, q_2)-*LDFOWGA*.

$\mathfrak{R}_1$	$(\langle 0.8395, 0.8724\rangle, \langle 0.6430, 0.3158\rangle)$
$\mathfrak{R}_2$	$(\langle 0.9497, 0.8881\rangle, \langle 0.4598, 0.4177\rangle)$
$\mathfrak{R}_3$	$(\langle 0.8384, 0.8607\rangle, \langle 0.5929, 0.2943\rangle)$
$\mathfrak{R}_4$	$(\langle 0.9397, 0.8375\rangle, \langle 0.5261, 0.2918\rangle)$

Table 9: (q_1, q_2)-*LDFHWGA*.

$\mathfrak{R}_1$	$(\langle 0.7881, 0.8978\rangle, \langle 0.5616, 0.3993\rangle)$
$\mathfrak{R}_2$	$(\langle 0.9215, 0.9119\rangle, \langle 0.398, 0.4938\rangle)$
$\mathfrak{R}_3$	$(\langle 0.8018, 0.8931\rangle, \langle 0.4638, 0.3945\rangle)$
$\mathfrak{R}_4$	$(\langle 0.9225, 0.8816\rangle, \langle 0.44427, 0.4182\rangle)$

Table 10: (q_1, q_2)-*LDFWAA* score values.

		$S_{(q_1,q_2)\text{-}LDFN}$		
	$S_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_1)$	$S_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_2)$	$S_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_3)$	$S_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_4)$
(q_1, q_2)-*LDFWAA*	0.193747	0.104891	0.210543	0.292625
(q_1, q_2)-*LDFOWAA*	0.201223	0.11617	0.245497	0.283939
(q_1, q_2)-*LDFHWAA*	0.101633	0.043234	0.123648	0.221248
		$\$_{(q_1,q_2)\text{-}LDFN}(\Upsilon)$		
	$\$_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_1)$	$\$_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_2)$	$\$_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_3)$	$\$_{(q_1,q_2)\text{-}LDFN}(\mathfrak{R}_4)$
(q_1, q_2)-*LDFWAA*	0.197124	0.168693	0.258444	0.344792
(q_1, q_2)-*LDFOWAA*	0.19712	0.186226	0.296881	0.334159
(q_1, q_2)-*LDFHWAA*	0.084448	0.099841	0.171944	0.268985

Table 10: Continued.

	$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_1)$	$\check{E}_{(q_1,q_2)\text{-}LDFN}$ $\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_2)$	$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_3)$	$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_4)$
(q_1,q_2)-$LDFWAA$	0.596874	0.552446	0.605271	0.646313
(q_1,q_2)-$LDFOWAA$	0.600612	0.558085	0.622748	0.64197
(q_1,q_2)-$LDFHWAA$	0.550816	0.521617	0.561824	0.610624

Table 11: (q_1,q_2)-$LDFWGA$ score values.

	$S_{(q_1,q_2)\text{-}LDFN}(\Re_1)$	$S_{(q_1,q_2)\text{-}LDFN}$ $S_{(q_1,q_2)\text{-}LDFN}(\Re_2)$	$S_{(q_1,q_2)\text{-}LDFN}(\Re_3)$	$S_{(q_1,q_2)\text{-}LDFN}(\Re_4)$
(q_1,q_2)-$LDFWGA$	0.124125	0.039068	0.102611	0.140018
(q_1,q_2)-$LDFOWGA$	0.147142	0.051862	0.138124	0.168274
(q_1,q_2)-$LDFHWGA$	0.242691	0.13481	0.188538	0.268998
	$\$_{(q_1,q_2)\text{-}LDFN}(\Re_1)$	$\$_{(q_1,q_2)\text{-}LDFN}$ $\$_{(q_1,q_2)\text{-}LDFN}(\Re_2)$	$\$_{(q_1,q_2)\text{-}LDFN}(\Re_3)$	$\$_{(q_1,q_2)\text{-}LDFN}(\Re_4)$
(q_1,q_2)-$LDFWGA$	0.119373	0.065824	0.0866	0.162998
(q_1,q_2)-$LDFOWGA$	0.129625	0.097895	0.111879	0.187492
(q_1,q_2)-$LDFHWGA$	0.222243	0.144246	0.19667	0.324315
	$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_1)$	$\check{E}_{(q_1,q_2)\text{-}LDFN}$ $\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_2)$	$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_3)$	$\check{E}_{(q_1,q_2)\text{-}LDFN}(\Re_4)$
(q_1,q_2)-$LDFWGA$	0.562063	0.519534	0.551305	0.570009
(q_1,q_2)-$LDFOWGA$	0.573571	0.525931	0.569062	0.584137
(q_1,q_2)-$LDFHWGA$	0.621346	0.567405	0.594269	0.634499

Table 12: Ranking of (q_1,q_2)-$LDFWAA$ operator w.r.t. $S_{(q_1,q_2)\text{-}LDFN}$, $\$_{(q_1,q_2)\text{-}LDFN}$ and $\check{E}_{(q_1,q_2)\text{-}LDFN}$.

	$S_{(q_1,q_2)\text{-}LDFN}$
(q_1,q_2)-$LDFWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
(q_1,q_2)-$LDFOWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
(q_1,q_2)-$LDFHWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
	$\$_{(q_1,q_2)\text{-}LDFN}$
(q_1,q_2)-$LDFWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
(q_1,q_2)-$LDFOWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
(q_1,q_2)-$LDFHWAA$	$\Re_4 > \Re_3 > \Re_2 > \Re_1$
	$\check{E}_{(q_1,q_2)\text{-}LDFN}$
(q_1,q_2)-$LDFWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
(q_1,q_2)-$LDFOWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$
(q_1,q_2)-$LDFHWAA$	$\Re_4 > \Re_3 > \Re_1 > \Re_2$

Table 13: Ranking of (q_1,q_2)-$LDFWGA$ operator w.r.t. $S_{(q_1,q_2)\text{-}LDFN}$, $\$_{(q_1,q_2)\text{-}LDFN}$ and $\check{E}_{(q_1,q_2)\text{-}LDFN}$.

	$S_{(q_1,q_2)\text{-}LDFN}$
(q_1,q_2)-$LDFWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
(q_1,q_2)-$LDFOWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
(q_1,q_2)-$LDFHWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
	$\$_{(q_1,q_2)\text{-}LDFN}$
(q_1,q_2)-$LDFWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
(q_1,q_2)-$LDFOWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
(q_1,q_2)-$LDFHWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
	$\check{E}_{(q_1,q_2)\text{-}LDFN}$
(q_1,q_2)-$LDFWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
(q_1,q_2)-$LDFOWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$
(q_1,q_2)-$LDFHWGA$	$\Re_4 > \Re_1 > \Re_3 > \Re_2$

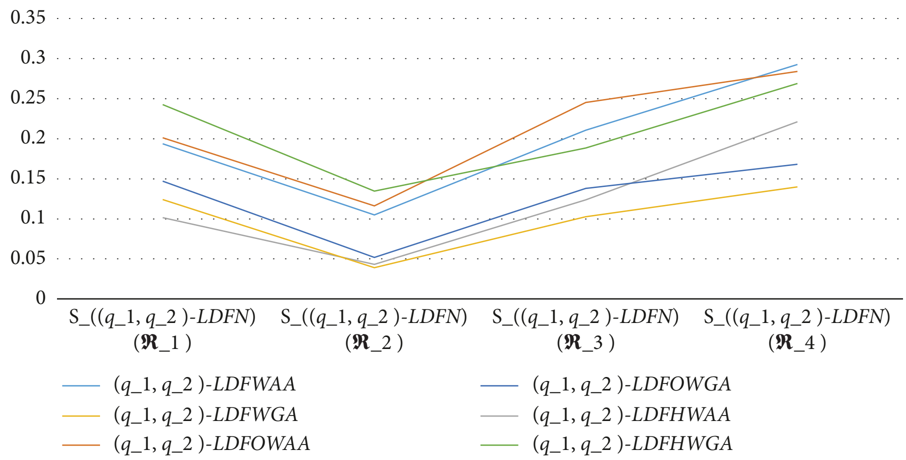

FIGURE 4: Scores of alternative based on the six aggregation operators.

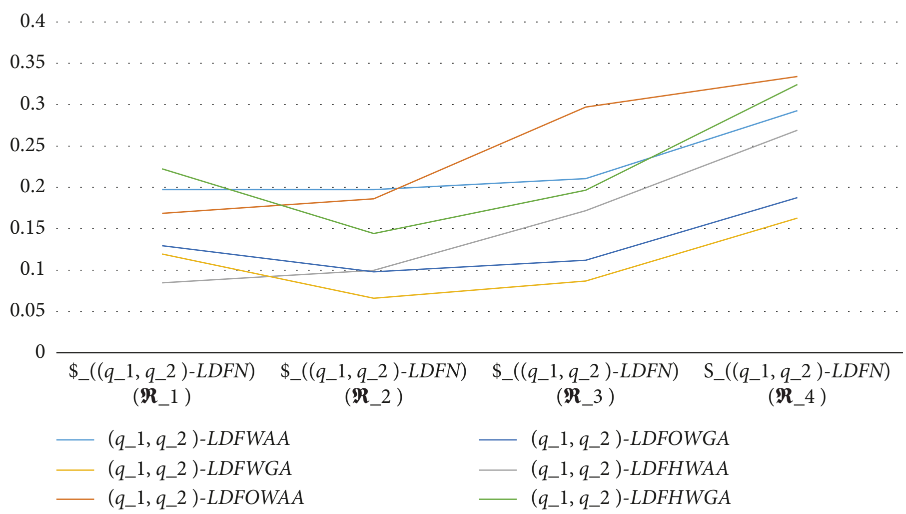

FIGURE 5: Quadratic scores of alternative based on the six aggregation operators.

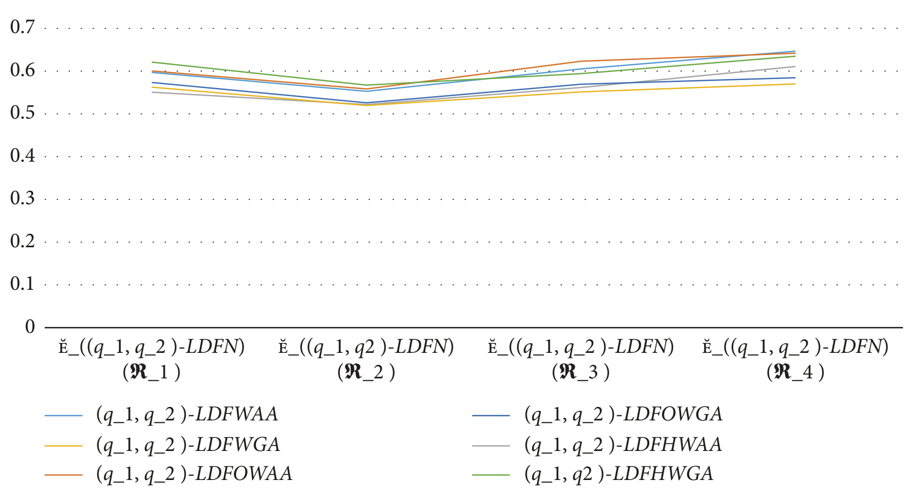

FIGURE 6: Expectation scores of alternative based on the six aggregation operators.

Table 14: Analyzing the intended work in comparison to the classical work.

Aggregations operators	References parameter mappings	Score functions
n,m-ROFWPA-operator [34]	No	Not calculable
LDFWGA-operator [7]	No	Not calculable
LDFEPWA-operator [10]	No	Not calculable
LDFEPWG-operator [10]	No	Not calculable
LDFWA-operator [31]	No	Not calculable
LDFWG-operator [31]	No	Not calculable
LDFEWA-operator [14]	No	Not calculable
LDFEOWA-operator [14]	No	Not calculable
LDFEWG-operator [14]	No	Not calculable
LDFEOWG-operator [14]	No	Not calculable
q-RLDFOWAA-operator [23]	No	Not calculable
q-RLDFHWAA-operator [23]	No	Not calculable
q-RLDFOWGA-operator [23]	No	Not calculable
q-RLDFHWGA-operator [23]	No	Not calculable
(p, q)RLDFWA-operator [26]	No	Not calculable
(p,q)RLDFGA-operator [26]	No	Not calculable
(q_1, q_2)-$LDFWAA$-operator	Yes	$\mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2$
(q_1, q_2)-$LDFOWAA$-operator	Yes	$\mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2$
(q_1, q_2)-$LDFHWAA$-operator	Yes	$\mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2$
(q_1, q_2)-$LDFWGA$-operator	Yes	$\mathfrak{R}_4 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_2$
(q_1, q_2)-$LDFOWGA$-operator	Yes	$\mathfrak{R}_4 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_2$
(q_1, q_2)-$LDFHWGA$-operator	Yes	$\mathfrak{R}_4 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_2$

Table 15: The distinctive analyses of various techniques.

Methods	Information correlation	Monotonicity	Flexibility	Deal with (q_1, q_2)-$LDFS$
n,m-ROFWPA-operator [34]	×	×	×	×
LDFWGA-operator [7]	×	×	×	×
LDFEPWA-operator [10]	×	×	×	×
LDFEPWG-operator [10]	×	×	×	×
LDFWA-operator [31]	×	×	×	×
LDFWG-operator [31]	×	×	×	×
LDFEWA-operator [14]	×	×	×	×
LDFEOWA-operator [14]	×	×	×	×
LDFEWG-operator [14]	×	×	×	×
LDFEOWG-operator [14]	×	×	×	×
q-RLDFOWAA-operator [23]	×	×	×	×
q-RLDFHWAA-operator [23]	×	×	×	×
q-RLDFOWGA-operator [23]	×	×	×	×
q-RLDFHWGA-operator [23]	×	×	×	×
(p,q)RLDFWA-operator [26]	×	×	×	×
(p,q)RLDFGA-operator [26]	×	×	×	×
(q_1, q_2)-$LDFWAA$-operator	✓	✓	✓	✓
(q_1, q_2)-$LDFOWAA$-operator	✓	✓	✓	✓
(q_1, q_2)-$LDFHWAA$-operator	✓	✓	✓	✓
(q_1, q_2)-$LDFWGA$-operator	✓	✓	✓	✓
(q_1, q_2)-$LDFOWGA$-operator	✓	✓	✓	✓
(q_1, q_2)-$LDFHWGA$-operator	✓	✓	✓	✓

- These developed operators, are exceptional cases. However, due to the ongoing ambiguity of decision data, these operators have some restrictions. Our improved operators are therefore far more efficient.

Please see Table 15 for a clearer comparison of the new method with the previous approaches.

7. Conclusion

An overview of the (q_1, q_2)- LDFS framework's expansion of all existing concepts and unrestricted foundation was given in the paper. The (q_1, q_2)- LDFS, an extension of the LDFSs that is enhanced in the membership space by combining it with the LDFS, was defined in formal terms. Under the (q_1, q_2)- LDFS, several aggregation operators and set theoretical procedures were created. The interesting properties of the proposed aggregating operators were examined. Additionally, a (q_1, q_2)-*LDF MADM* method was constructed with suggested scoring functions and aggregating operators. A case study was supplied to show how the suggested technique should be used. As a limitation on our research to demonstrate the viability of the suggested methodology, we only take into account four possibilities. Where the n, m-ROFSs, LDF number, did not function, the recommended technique does. While (q_1, q_2)-LDFS for both $q_1: J_1 \longrightarrow [0, 1]$, and $q_2: J_2 \longrightarrow [0, 1]$, the LDFS is only effective for $q_1(\alpha) = \alpha$ and $q_2(\beta) = \beta$. Comparing the suggested methodology to some of the existing MADM techniques, it has the following noteworthy successes: (1) The notion of parameter mappings $q_1: J_1 \longrightarrow [0, 1]$, and $q_2: J_2 \longrightarrow [0, 1]$ is about creating and managing a structured relationship between parameters across different systems or processes. This concept is vital in ensuring that data remains consistent, correctly interpreted, and accurately transformed as it moves through different parts of a system or between different systems; (2) defining a large information space with flexible membership grades and nonmembership grade constraints; (3) covering a variety of diverse operations; and (4) reducing the detrimental effects of excessive evaluation values made by DM and dynamically adjusting the weights assigned to input arguments. The provided strategy's methodology can be implemented as a computer program, allowing us to conduct our research for a restricted set of features and options while utilizing massive amounts of data and accounting for additional factors. Future studies should look into other aggregating operators like Hamacher and Bonferroni as well as expanding the recommended operators to the Archimedean norm.

Data Availability Statement

Data sharing is not applicable to this article, as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

Conceptualization, M.B.K. and A.M.D.; validation, A.M.D. and J.T.; formal analysis, A.M.D., J.T., M.V.C., and N.Z.; investigation, M.B.K., and B.B.-M.; resources, M.B.K., B.B.-M. and M.V.C.; writing – original draft, M.B.K., J.T., and B.B.-M.; writing – review and editing, M.B.K., A.M.D., and N.Z.; visualization, M.B.K., B.B.-M., and N.Z.; supervision, M.B.K., B.B.-M., and M.V.C.; project administration, M.B.K., B.B.-M., and J.T. All authors have read and agreed to the published version of the manuscript.

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