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Analytic study of a rolling sphere on a rough surface

Olivia A. Florea[1,a] and Ileana C. Rosca[2]
[1]*Department of Mathematics and Computer Science, Transilvania University of Brasov, Romania*
[2]*Department of Product Design, Mechatronics and Environment, Transilvania University of Brasov, Romania*

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In this paper it is realized an analytic study of the rolling's sphere on a rough horizontal plane under the action of its own gravity. The necessities of integration of the system of dynamical equations of motion lead us to find a reference system where the motion equations should be transformed into simpler expressions and which, in the presence of some significant hypothesis to permit the application of some original methods of analytical integration. In technical applications, the bodies may have a free rolling motion or a motion constrained by geometrical relations in assemblies of parts and machine parts. This study involves a lot of investigations in the field of tribology and of applied dynamics accompanied by experiments. Multiple recordings of several trajectories of the sphere, as well as their treatment of images, also followed by statistical processing experimental data allowed highlighting a very good agreement between the theoretical findings and experimental results. [http://dx.doi.org/10.1063/1.4967509]

I. INTRODUCTION

A very important problem that appears in many technical applications is the dynamical behavior of the bodies that are in a rolling motion, in real loading conditions and in connection with the environment.

This study involves a lot of investigations in the field of tribology and of applied dynamics accompanied by experiments. The characteristics of the dried friction in the mechanical systems were the subject of some independent studies from dynamics and tribology.[1]

In technical applications, the bodies may have a free rolling motion (e.g. the suppliers from the components of industrial automation of manufacture, fitting, control, etc.) or a motion constrained by geometrical relations in assemblies of parts and machine parts (e.g. rolling bearings, ball screws, etc.).[2]

Under the charge, the bodies are elastically and sometimes plastically deformed, obtaining contact surfaces whose dimensions are smaller than the dimensions of the contact bodies. These contacts form the friction couples.[3]

Ishkhanyan, in the paper,[4] proved that, for any initial conditions, the angular velocity of the rolling of the sphere and the sliding velocity vanish after the same finite time. It is shown that the sliding and rolling are interconnected and, in particular, that the rolling of a sphere without sliding is impossible. Also, he analyzes, in Ref. 5, the dynamics of a homogeneous ball on a horizontal plane with friction of all kinds, namely, sliding, spinning, and rolling friction, taken into account.

The problem on rolling of a homogeneous ball on an arbitrary surface was studied in Ref. 6. In our paper we try to simplify the theoretical study that leads to differential equations very complicated with solutions very difficult to exploit. Therefore we will neglect one of the components of the instantaneous rotation velocity.

[a]Corresponding author: olivia.florea@unitbv.ro

Let a spherical body with the radius *R* be considered into a motion on a fixed horizontal surface under its own gravity which has the property that it continuously keeps the contact with the surface. In the figure below (fig. 1) are represented the fixed and mobile coordinate systems.

In report with the classical definition of Euler's angles, we may consider another two main directions: ON and OT. These directions with the Oz axis can form an orthonormal trihedral. This trihedral could be the base of another coordinates system, knowing under the name of Resal trihedral. This coordinates system leads to some simpler expressions for the components of the instantaneous rotation velocity.

To determine the angular velocity of a sphere during the rolling motion, we will consider the same configuration like in the figure 1 and we will define a new mobile coordinate system which is moving jointly with the sphere – the Resal coordinates system which has the axis: *ON* with the unit vector $\vec{e}_1$, *OT* with the unit vector $\vec{e}_2$ and *Oz* with the unit vector $\vec{k}$. In this context, the point *P* is in the plane (*zOT*) because *P* is on the Oz_1 axis and the plane (*zOT*) coincides with the plane (zOz_1), the axis *OT* being the intersection between the plane (zOz_1) and the mobile plane (*xOy*). Reciprocally, we can state that the plane (*zOT*) follows the motion of the point *P*.

Analyzing the motion of a sphere launched on a rough horizontal plane, we can draw the conclusion that this rolls around his axis, slides on this plane and performs a nutation and precession motions. It is supposed that the sphere is under the influence of its own gravity and therefore the friction reactions of the contact plane will be derived: the normal reaction and the slide friction force, neglecting the rolling friction and the yaw friction.

The rotation vector $\vec{\omega}$ of the sphere and the angular velocity vector $\vec{\Omega}$ of the mobile coordinate system *ONTz,* which permanently accompanies the sphere in motion, have the form:

$$\vec{\omega} = \vec{\dot{\varphi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}} = \dot{\theta} \cdot \vec{e}_1 + \dot{\psi} \cdot sin\theta \cdot \vec{e}_2 + \left(\dot{\psi} \cdot cos\theta + \dot{\varphi}\right)\vec{k},, \tag{1}$$

$$\vec{\Omega} = \vec{\dot{\theta}} + \vec{\dot{\psi}} = \dot{\theta} \cdot \vec{e}_1 + \dot{\psi} \cdot sin\theta \cdot \vec{e}_2 + \dot{\psi} \cdot cos\theta\,\vec{k}, \tag{2}$$

ψ, φ, θ are the precession, proper rotation and nutation angles.

If the angular velocity of the sphere is big enough, then the variation of nutation can be neglected: $\dot{\theta} = 0 \Leftrightarrow \theta = \theta_o =$ constant.

To determine the motion of the point *P* we use the expression of its position vector:

$$\vec{r} = \overrightarrow{OP} = -R \cdot sin\theta_o \cdot \vec{e}_2 - R \cdot cos\theta_o\,\vec{k} \tag{3}$$

We can apply the theorem of the impulse variation and the theorem of the variation of the kinetic momentum in the specific conditions of the studied configuration.

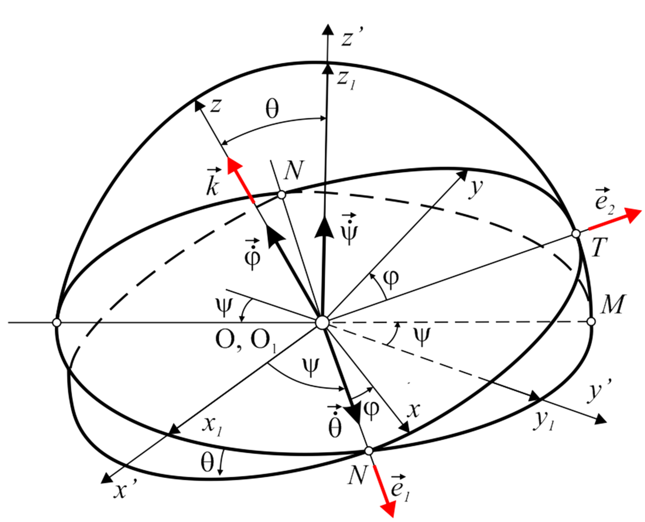

FIG. 1. The coordinate system of the body found in a motion on a fixed surface.

From the theorem of the impulse variation:

$$M \cdot \frac{d\vec{v}_o}{dt} = M\vec{g} + \vec{N} + \vec{\Phi}, \tag{4}$$

where $\vec{\Phi} = -\frac{\mu M g}{v_P}\left(v_{P_{e1}} \cdot \vec{e}_1 + v_{P_{e2}} \cdot \vec{e}_2 + v_{P_{e3}} \cdot \vec{k}\right)$ is the friction force and in the conditions when the proper gravity $M\vec{g}$ and the normal reaction $\vec{N}$ of the support surface are in equilibrium $M\vec{g} + \vec{N} = 0$, after some computations we obtain three scalar equations:

$$\begin{cases} \dot{v}_{O_{\vec{e}1}} - \dot{\psi} \cdot cos\theta_o \cdot \dot{v}_{O_{\vec{e}2}} + \dot{\psi} \cdot sin\theta_o \cdot \dot{v}_{O_{\vec{k}}} = -\frac{\mu g}{v_P} v_{P_{\vec{e}1}}; \\ \dot{\psi} \cdot cos\theta_o \cdot \dot{v}_{O_{\vec{e}1}} + \dot{v}_{O_{\vec{e}2}} = -\frac{\mu g}{v_P} v_{P_{\vec{e}2}}; \\ -\dot{\psi} \cdot sin\theta_o \cdot \dot{v}_{O_{\vec{e}1}} + \dot{v}_{O_{\vec{k}}} = -\frac{\mu g}{v_P} v_{P_{\vec{k}}}. \end{cases} \tag{5}$$

From the theorem of the variation of the kinetic momentum in report with the sphere centroid

$$\frac{d\vec{K}_o}{dt} = \vec{M}_o, \tag{6}$$

II. DETERMINING OF EQUATIONS OF MOTION

The sphere's centrum has the coordinates ξ, η, ζ in report with the fixed system $Ox'y'z'$ and the forces that act on it are: its own gravity $M\vec{g}$, the sliding friction force or the adhesion force $\vec{\Phi}$ and the reaction force normal to the surface $\vec{N}$. In the conditions when the torques of yaw friction and those of the rolling friction are neglected, the motion equations are written using the theorems for the torsor's impulses. The condition that the sphere slides on the horizontal plane is:

$$\vec{\Phi} = \mu \ \vec{N} \ , \tag{7}$$

The equations of motion corresponding to the theorem of the motion of the mass centrum will be in the fixed coordinate system:

$$\begin{cases} M \cdot \ddot{\xi} = X; \\ M \cdot \ddot{\eta} = Y; \\ M \cdot \ddot{\zeta} = 0 = -Mg + N, \end{cases} \tag{8}$$

where we note by: $\vec{\Phi}(X, Y, 0)$ for the slide friction and $\vec{N}(0, 0, N)$ for the normal reaction of the plane. From the theorem of the kinetic momentum we have:

$$\vec{K}_o = \overline{\overline{J}} \cdot \vec{\omega}, \tag{9}$$

where the shape particularities of the sphere are taken into account:

$$\begin{aligned} &J_{xx} = J_{yy} = J_{zz} = \tfrac{2}{5} M R^2; \\ &J_{xy} = J_{yx} = J_{yz} = J_{zy} = J_{xz} = J_{zx} = 0; \\ &J_o = \tfrac{1}{2}\left(J_{xx} + J_{yy} + J_{zz}\right) = \tfrac{3}{5} M R^2. \end{aligned} \tag{10}$$

In this manner we obtain a second group of equations:

$$\begin{cases} \frac{2}{5} M R^2 \cdot \dot{\omega}_{x_1} = RY; \\ \frac{2}{5} M R^2 \cdot \dot{\omega}_{y_1} = -RX; \\ \frac{2}{5} M R^2 \cdot \dot{\omega}_{z_1} = 0. \end{cases} \tag{11}$$

Using the Euler relation for the velocity distribution, the velocity at the contact point P of the sphere is: $\vec{v}_P = \vec{v}_o + \vec{\omega} \times \overline{OP}$. Based on the previous results, we obtain the vector form:

$$\dot{\vec{v}}_P = \frac{7}{2M}\vec{\Phi}. \tag{12}$$

During the slide, the velocity at the contact point is collinear with the friction or adhesion force. We consider a proportionality factor that is a positive scalar depending on time:

$$\vec{\Phi} = -\lambda^2(t) \cdot \vec{v}_P. \tag{13}$$

Further we deduce the new form of the equation (6):

$$\frac{d\vec{v}_P}{dt} = -\frac{7}{2M}\lambda^2(t) \cdot \vec{v}_P, \tag{14}$$

which is equivalent with two scalar equations that can be integrated function of time, and after some successive transformations, the slide velocity component at the contact point are derived:

$$\begin{matrix} v_{P_x} = v_{P_{xo}} \cdot e^{-\frac{7}{2M}\int_{t_o}^{t} \lambda^2(t)\,dt} \\ v_{P_y} = v_{P_{yo}} \cdot e^{-\frac{7}{2M}\int_{t_o}^{t} \lambda^2(t)\,dt} \end{matrix} \Rightarrow v_P = e^{-\frac{7}{2M}\sqrt{v_{P_{xo}}^2 + v_{P_{yo}}^2}\int_{t_o}^{t} \lambda^2(t)\,dt}. \tag{15}$$

Analyzing the obtained results we can highlight some conclusions regarding the sphere dynamics which slides on the horizontal plane:

- Because the velocity component in the point *P*, have in every moment proportional values to its modulus, it results that $\vec{v}_P$ has a fixed direction;
- The velocity $\vec{v}_P$ decreases in time;
- If $\vec{v}_P$ has fixed direction and taking into account the relation between the velocity and the friction force, we can make the remark that the friction force has the modulus and the direction constant;
- The velocity components $\vec{v}_P$ after the axis from the slide plane v_{P_x} and v_{P_y} are time-linear functions;
- Because $\vec{v}_P$ has a fixed direction and because v_{P_x} and v_P are time-linear functions, it results that the ratio v_P/v_{P_x} is constant, which means that v_{P_x} and v_P become null in the same time, that leads to the fact that the slide could stop at a specified moment.

With the kinetic momentum $\vec{K}_o = \overline{\overline{J}} \cdot \vec{\omega}$, $J = J_{xx} = J_{yy} = J_{zz} = \frac{2}{5}MR^2$ and for the momentum of the external forces given by $\vec{\boldsymbol{M}}_o = \overrightarrow{OP} \times \vec{N} + \overrightarrow{OP} \times \vec{\Phi}$, after some computations we obtain another three scalar equations:

$$J\dot{\varphi}\dot{\psi} \cdot sin\theta_o = \mu MgR\left(-_o \cdot v_{P_{\vec{e}1}} + sin\theta_o \cdot v_{P_{\vec{k}}}\right)/v_P; \tag{16a}$$

$$J\ddot{\psi} \cdot sin\theta_o = \mu MgR \cdot cos\theta_o \cdot v_{P_{\vec{e}1}}/v_P; \tag{16b}$$

$$J\left(\ddot{\psi} \cdot cos\theta_o + \ddot{\varphi}\right) = -\mu MgR \cdot \sin\theta_o \cdot v_{P_{\vec{e}1}}/v_P. \tag{16c}$$

Due to the fact that the sphere should stay in permanent contact with the plane, it results that its mass centrum couldn't move on the vertical direction, in our case the Oz_1 axis, fact that gives us the conditions:

$$\begin{aligned} v_{P_{\vec{k}1}} &= v_{P_{\vec{e}2}} \cdot sin\theta_o + v_{P_{\vec{k}}} \cdot cos\theta_o = 0; \\ v_{O_{\vec{k}1}} &= v_{O_{\vec{e}2}} \cdot sin\theta_o + v_{O_{\vec{k}}} \cdot cos\theta_o = 0. \end{aligned} \tag{17}$$

Using these conditions, after some computations we obtain a new form of the equation (16):

$$J\dot{\varphi}\dot{\psi} \cdot sin\theta_o = -\left(\mu MgR \cdot v_{P_{\vec{e}2}} \cdot sec\theta_o\right)/v_P. \tag{18}$$

Multiplying the equation (16b) with $sin\theta_0$ and (16c) with $cos\theta_0$ and then adding the obtained relations we will have the differential equation:

$$J\ddot{\psi} + J\ddot{\phi}cos\theta_0 = 0 \tag{19}$$

therefore we have:

$$\dot{\psi} + \dot{\phi}cos\theta_0 = C_1 \tag{20}$$

The equation (20) highlight that the kinetic momentum of the sphere across the Oz_1 axis is conserved due to the external forces that act on it. These forces can be parallel with the axis $Oz_1(M\vec{g}, \vec{N})$, or can intersect this axis $\vec{\Phi}$. In analogous mode multiplying the equation (16b) with $cos\theta_0$ and (16c) with $-sin\theta_0$ and after that adding the obtained equations we have:

$$-J\ddot{\phi}sin\theta_0 = \mu MGR\frac{v_{P\vec{e}_1}}{v_P} \tag{21}$$

Raising on square the relation (21) and adding the result at the square of the relation (18) we have:

$$J^2\ddot{\phi}\,sin^2\,\theta_0 + J^2\dot{\phi}^2\dot{\psi}^2\,sin^2\,\theta_0 = (\mu MgR)^2\frac{v^2_{P\vec{e}_1} + v^2_{P\vec{e}_2}\sec^2\theta_0}{v_P^2} \tag{22}$$

III. OBTAINING OF THE SOLUTIONS

Because

$$v_P^2 = v^2_{P\vec{e}_1} + v^2_{P\vec{e}_2}\,sec^2\,\theta_0 \tag{23}$$

The relation (22) will be:

$$J^2(\ddot{\phi}^2 + \dot{\phi}^2\dot{\psi}^2)\,sin^2\,\theta_0 = (\mu MgR)^2 \tag{24}$$

Based on (20), the equation (24) is reduced to a differential equation in the unknown ϕ:

$$J^2\ddot{\phi}^2\,sin^2\,\theta_0 + J^2\dot{\phi}^2\,(C_1 - \dot{\phi}cos\theta_0)^2\,sin^2\,\theta_0 = (\mu MgR)^2$$

The above relation can be written in the following form:

$$J\ddot{\phi}sin\theta_0 = \pm\sqrt{a^2 + e^2}\cdot\sqrt{b^2 - e^2} \tag{25}$$

where:

$$e = \dot{\phi}\sqrt{\frac{J}{2}sin2\theta} - \frac{C_1}{2}\sqrt{J\mathrm{tg}\theta_0}$$

$$a = \sqrt{\mu MgR - J\frac{C_1^2}{4}\mathrm{tg}\theta_0}$$

$$b = \sqrt{\mu MgR + J\frac{C_1^2}{4}\mathrm{tg}\theta_0}$$

Deriving the relation (26) in report with time and replacing in (25) we obtain the differential equation with separable variables:

$$\sqrt{J\mathrm{tg}\theta_0}\dot{e} = \pm\sqrt{(a^2 + e^2)(b^2 - e^2)} \tag{26}$$

Further, within the integration we obtain:

$$\int_0^b \frac{de}{\sqrt{(a^2 + e^2)(b^2 - e^2)}} = \int_0^t \frac{dt}{\sqrt{J\mathrm{tg}\theta_0}} \tag{27}$$

Note by: $I = \int_0^b \frac{de}{\sqrt{(a^2+e^2)(b^2-e^2)}}$ and using the change variable $e = b\cdot sinx$ and noting $k = b/a$ the integral will be: $I = k\int_0^{\frac{\pi}{2}} F(k,x)dx$,[7] where $F(k,x) = \frac{1}{\sqrt{a+k^2\,sin^2\,x}}$. Using the series development, because $|k|<1$, the power series is uniform and absolute convergent on $\left(0, \frac{\pi}{2}\right)$ that permits the integration term by term:

$$F(k,x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} k^{2n} \sin^{2n} x$$

After the integration of the terms from $F(k,x)$ we obtain the recurrence relation:

$$I_{2n} = \frac{2n-1}{2n-2} I_{2n-2}, \text{ where} I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx$$

Therefore we have:

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \cdot \frac{\pi}{2}$$

Hence, the integral from the right member of (30) will be:

$$I = \frac{k\pi}{2} \left[1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{(2n)!}{2^{2n}(n!)^2} \right)^2 k^{2n} \right]$$

Using the trigonometric form of the complex number and the Euler's formula we obtain $k = e^{-i\theta_0}$, thus the integral I becomes:

$$I = \frac{e^{-i\theta_0}}{2} \left[1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{(2n)!}{2^{2n}(n!)^2} \right)^2 e^{-2in\theta_0} \right]$$

If $\theta < \frac{\pi}{2}$ the value of I in first approximation will be:

$$I = \frac{3\pi}{8}$$

Returning to the equation (27), we can determine the approximate value of t:

$$t = t_0 \pm \frac{3\pi}{8} \sqrt{J \text{tg} \theta_0} \tag{28}$$

IV. RESULTS AND DISCUSSIONS

The slide velocity of the sphere on the rough surface can be described by the relation:

$$v_P = v_{P_0} - \frac{7}{2} \mu g t \tag{29}$$

from where we can draw the conclusion that

$$t > t_0 = \frac{2}{7} \frac{v_{P_0}}{\mu g} \tag{30}$$

In Ref. 8 using the experimental determination of the oscillation period it was found that the period takes values presented in the Table I.

Considering that the radius of the sphere is R=17mm, and μ=0.8, based on (30) we obtain that $t_0 = 0.07$. Based on (28) we obtain that t=0.074.

TABLE I. Variation of turation and angular seed.

Period	Turation [rot/min]	Anguar speed[rad/s]
(0.102; 0.168)	531.76	56
(0.079; 0.137)	1580	165
(0.058; 0.123)	2080.53	218

The slide velocity of the sphere on the surface is those of its contact point P with the rough trajectory. Transforming the equations (16) by taking into account the further conditions (17) and the derivative of the P point speed on time, we arrive at the condition:

$$\dot{v}_P = \frac{dv_P}{dt} = -\frac{7}{2}\mu g,$$

where from, by integrating:

$$\int_{v_{P_0}}^{v_P} dv_P = -\frac{7}{2}\mu g \int_0^t dt, \Rightarrow v_P = v_{P_0} - \frac{7}{2}\mu g t. \tag{31}$$

From the above relation results the next conclusion: there is a time limit when v_P cannot be negative (considering the fact that P is always in the rolling plan).

When

$$t > t_0' = \frac{2}{7}\frac{v_{P_0}}{\mu g}, \tag{32}$$

the speed v_P becomes zero, and beginning with this moment the movement evolves towards an inertial rotation or a pure rolling.

A particular case is those when $t \le t_0'$, when from conditions (17), (18), and considering $J = \frac{2}{5}MR_2$ the inertial momentum of the sphere, we shall obtain the sphere center speed:

$$v_{O_{\vec{e}_1}} = v_{P_{\vec{e}_1}} - R\dot{\varphi}\sin\theta_0. \tag{33}$$

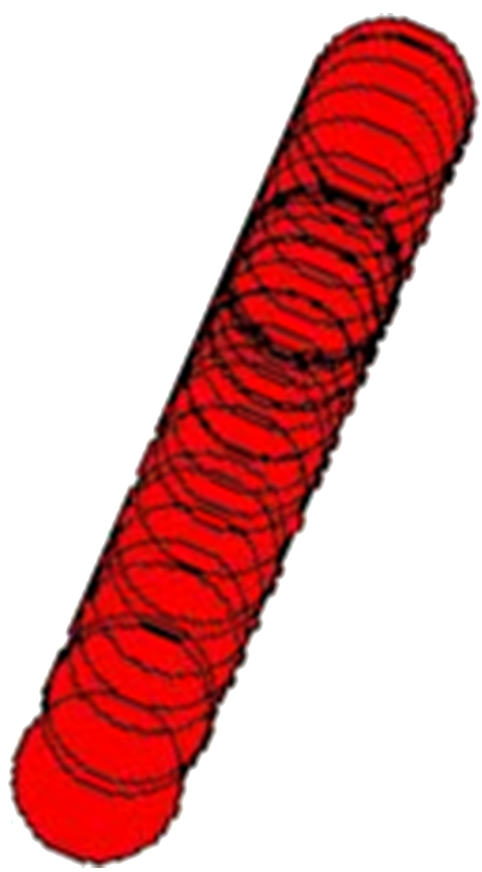

FIG. 2. The images of the frames overlaid on the firs frame.

On the other side, from equation (16b), we can obtain after transformations:

$$v_{P_{\vec{e}_1}} = -\frac{2}{5}\frac{R}{\mu g}\ddot{\varphi}\sin\theta_0 \cdot v_P. \tag{34}$$

Replacing (32) in (31) we shall obtain the expression of mass center of the sphere:

$$v_{O_{\vec{e}_1}} = -R\dot{\varphi}\sin\theta_0 - \frac{2}{5}\frac{R}{\mu g}\ddot{\varphi}\sin\theta_0 \cdot v_P \tag{35}$$

that allows to express the mass center speed components in a detailed form, analogous to the contact point speed v_P, i.e. as a pseudo periodical functions Jacobi, by an elliptic cosine.

The experiment consists in identification of the measurements that highlight the studied phenomenon, the determination of the conditions, the design of the experimental plant and the method of obtaining the experimental data. To highlight the periodicity of the instantaneous rotation velocity, we decided to verify the periodicity of a coordinates system with the shape of a meridian traced on a sphere in rolling motion, and which, in the launch momentum on the rough horizontal surface, has a vertical position along the launch direction (fig2).

V. CONCLUSIONS

This theoretical study and its conclusions are based on the hypothesis that for an enough great rotational speed, the nutation variation can be neglected in report to others speed components.

The arising question would be: the real nutation variation could it hardly influence the theoretical results as to make them irrelevant? Thus it is imperatively necessary to experimentally verify the theoretical results.

As for $\dot{\theta} = 0$, $\dot{\psi}$ and $\dot{\varphi}$ are of time periodic through a cosine elliptic function, the instantaneous rotational speed in periodical during a certain laps of time. Experimental tests were established in order to proof this behavior – the movement periodicity of a meridian mark drawn on the sphere, and having at the beginning a vertical position, along the direction of travel. If theoretical conclusions are verified, then the mark will perform a periodical movement from one side to the other of its initial position. Consequently, it will be necessary to show, during the experiment, successive positions of the meridian mark oscillating on the two sides of its initial position.

As the studied phenomenon was in real time, a visual qualitative method was chosen for analyzing of the continuous successive positions. The phenomenon was observed at a higher speed than the human eye discriminating speed of moving body successive positions. Further the images were treated by filtration, reduction to contours, rescaling and reassembled in order to highlight the periodicity of the position of the marker and the straight trajectory.

Multiple recordings of several trajectories of the sphere, as well as their treatment of images, also followed by statistical processing experimental data allowed highlighting a very good agreement between the theoretical findings and experimental results.

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