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Some Improvements of the Cauchy-Schwarz Inequality Using the Tapia Semi-Inner-Product

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Abstract: The aim of this article is to establish several estimates of the triangle inequality in a normed space over the field of real numbers. We obtain some improvements of the Cauchy–Schwarz inequality, which is improved by using the Tapia semi-inner-product. Finally, we obtain some new inequalities for the numerical radius and norm inequalities for Hilbert space operators.

Keywords: inner product space; triangle inequality; Cauchy–Schwarz inequality

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1. Introduction

In an inner product space an important inequality is the inequality of Cauchy–Schwarz [1,2], namely:

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{1}$$

for all $x, y \in X$, where X is a complex inner product space.

Aldaz [3] and Dragomir [4] studied the Cauchy–Schwarz inequality in the complex case.

Another inequality that plays a central role in a normed space is the triangle inequality,

$$\|x+y\| \leq \|x\| + \|y\|, \tag{2}$$

for all $x, y \in X$, where X is a complex normed space. Pečarić and Rajić in [5] proved other results about the triangle inequality.

In [6], Maligranda showed the following inequality:

$$A \cdot \min\{\|x\|, \|y\|\} \leq \|x\| + \|y\| - \|x+y\| \leq A \cdot \max\{\|x\|, \|y\|\}, \tag{3}$$

where $A = \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)$, and x and y are nonzero vectors in a normed space $X = (X, \|\cdot\|)$. Using this inequality we obtain an estimate for the *norm-angular distance* or *Clarkson distance* (see, e.g., [7]) between nonzero vectors x and y, $\alpha[x, y] = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$, thus [8]:

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \alpha[x, y] \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}. \tag{4}$$

In [6], Maligranda generalized the norm-angular distance to the p-angular distance in a normed space given by:

$$\alpha_p[x,y] := \| \|x\|^{p-1}x - \|y\|^{p-1}y\|,$$

where $p \in \mathbb{R}$. In [9], Dragomir studied new bounds for this distance. Other results for bounds for the angular distance, named Dunkl–Williams type theorems (see [10]), are given by Moslehian et al. [11] and Krnić and Minculete [12,13].

Dehghan [14] presented a new refinement of the triangle inequality and defined the skew angular distance between nonzero vectors x and y by $\beta[x,y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$. In [15], we remarked on several estimates of the triangle inequality using integrals and in [16] a characterization is given for a generalized triangle inequality in normed spaces.

The aim of this article is to establish several estimates of the triangle inequality in a normed space over the field of real numbers. This study is presented in Section 2. We also obtain, in Section 3, some improvements of the Cauchy–Schwarz inequality, which is improved by using the Tapia semi-inner-product. In Section 4, we obtain some new inequalities for the numerical radius and norm inequalities for Hilbert space operators.

2. Inequalities Related to the Triangle Inequality

We present some results regarding the several estimates of the triangle inequality in a normed space over the field of real numbers $\mathbb{R}$.

Theorem 1. *If $X = (X, \|\cdot\|)$ is a normed vector space over the field $\mathbb{R}$, then the following inequality holds*

$$\min\{\frac{a}{c}, \frac{b}{d}\}(c\|x\| + d\|y\| - \|cx+dy\|) \le a\|x\| + b\|y\| - \|ax+by\| \\ \le \max\{\frac{a}{c}, \frac{b}{d}\}(c\|x\| + d\|y\| - \|cx+dy\|), \quad (5)$$

for all vectors x and y in X and $a, b, c, d \in \mathbb{R}_+, c, d \neq 0$.

Proof. Without loss of generality, we may assume that $0 \le \frac{a}{c} \le \frac{b}{d}$. Then, by calculations and using the triangle inequality, we have

$$a\|x\| + b\|y\| - \min\{\frac{a}{c}, \frac{b}{d}\}(c\|x\| + d\|y\| - \|cx+dy\|) =$$

$$a\|x\| + b\|y\| - a\|x\| - \frac{ad}{c}\|y\| + \|ax + \frac{ad}{c}y\|) =$$

$$(b - \frac{ad}{c})\|y\| + \|ax + \frac{ad}{c}y\| = \|(b - \frac{ad}{c})y\| + \|ax + \frac{ad}{c}y\| \ge \|ax+by\|.$$

Therefore, we obtain the first inequality of the statement.

In the same way, according to inequality $\frac{a}{c} \le \frac{b}{d}$, we have $(\frac{bc}{d} - a)\|x\| = \|\frac{bc}{d}x - ax\|$ and we deduce

$$\max\{\frac{a}{c}, \frac{b}{d}\}(c\|x\| + d\|y\| - \|cx+dy\|) + \|ax+by\|$$

$$= \frac{bc}{d}\|x\| + b\|y\| - \|\frac{bc}{d}x + by\| + \|ax+by\|$$

$$= a\|x\| + b\|y\| + \left(\frac{bc}{d} - a\right)\|x\| + \|ax+by\| - \|\frac{bc}{d}x + by\|$$

$$= a\|x\| + b\|y\| + \|\frac{bc}{d}x - ax\| + \|ax + by\| - \|\frac{bc}{d}x + by\| \geq a\|x\| + b\|y\|.$$

Because, using the triangle inequality, we have

$$\|\frac{bc}{d}x - ax\| + \|ax + by\| \geq \|\frac{bc}{d}x + by\|,$$

for all vectors x and y in X and $a, b, c, d \in \mathbb{R}_+, c, d \neq 0$.

For $0 \leq \frac{b}{d} \leq \frac{a}{c}$, we make the similar calculations. Therefore, the inequalities of the statement are true. □

Remark 1. *(a) If $c = d$ in inequality (5), then we have*

$$\min\{a, b\}(\|x\| + \|y\| - \|x + y\|) \leq a\|x\| + b\|y\| - \|ax + by\| \leq \max\{a, b\}(\|x\| + \|y\| - \|x + y\|), \tag{6}$$

for all vectors x and y in X and $a, b \in \mathbb{R}_+$. Another extension of this result can be found in [13].

(b) For $x \to \frac{x}{\|x\|}$ and $y \to \frac{y}{\|y\|}$ in relation (6), we obtain the following inequalities:

$$\min\{a, b\}\Big(2 - \Big\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\Big\|\Big) \leq a + b - \Big\|\frac{a}{\|x\|}x + \frac{b}{\|y\|}y\Big\| \leq \max\{a, b\}\Big(2 - \Big\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\Big\|\Big), \tag{7}$$

for all nonzero vectors x and y in X and $a, b \in \mathbb{R}_+$. For $a = \|x\|$ and $b = \|y\|$ in inequality (7), we find inequality (3).

(c) For the nonzero vectors x and y in X, in Theorem 1, we make the following substitutions: $a = d = \frac{1}{\|x\|}, b = c = \frac{1}{\|y\|}$. We obtain

$$\min\{\frac{\|x\|}{\|y\|}, \frac{\|y\|}{\|x\|}\}\Big(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \Big\|\frac{x}{\|y\|} + \frac{y}{\|x\|}\Big\|\Big) \leq 2 - \Big\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\Big\| \leq \max\{\frac{\|x\|}{\|y\|}, \frac{\|y\|}{\|x\|}\}\Big(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \Big\|\frac{x}{\|y\|} + \frac{y}{\|x\|}\Big\|\Big). \tag{8}$$

In relation (8), if we replace y by $-y$, then we have an inequality which uses the angular distance and the skew angular distance, thus:

$$\min\{\frac{\|x\|}{\|y\|}, \frac{\|y\|}{\|x\|}\}\Big(\frac{(\|x\| - \|y\|)^2}{\|x\|\|y\|} - \beta[x, y]\Big) \leq 2 - \alpha[x, y] \leq \max\{\frac{\|x\|}{\|y\|}, \frac{\|y\|}{\|x\|}\}\Big(\frac{(\|x\| - \|y\|)^2}{\|x\|\|y\|} - \beta[x, y]\Big), \tag{9}$$

for all nonzero vectors x and y in X.

Theorem 2. *If $X = (X, \|\cdot\|)$ is a normed vector space over the field of real numbers $\mathbb{R}$, with $\|x\| = \|y\| = 1$, then we have*

$$\frac{\Big\|ax + by\Big\| - |a - b|}{\min\{a, b\}} \leq \Big\|x + y\Big\| \leq \frac{\Big\|ax + by\Big\| + |a - b|}{\max\{a, b\}} \tag{10}$$

for every vector x and y in X and $a, b \in \mathbb{R}_+^*$.

Proof. From relation (6), using the first inequality, we find,

$$a+b-|a-b|-\min\{a,b\}\left\|x+y\right\| \le a+b-\left\|ax+by\right\|,$$

which implies

$$\left\|ax+by\right\|-|a-b| \le \min\{a,b\}\left\|x+y\right\|.$$

Using the second inequality of relation (6), we find the inequality

$$a+b-\left\|ax+by\right\| \le a+b+|a-b|-\max\{a,b\}\left\|x+y\right\|,$$

which becomes

$$\max\{a,b\}\left\|x+y\right\| \le \left\|ax+by\right\|+|a-b|,$$

for all vectors x and y in X and $a, b \in \mathbb{R}_+^*$.

Therefore, we deduce the relation of the statement. □

Remark 2. *If we replace y by $-y$ in Theorem 2, then we obtain an inequality for Clarkson distance, in the case $\|x\| = \|y\| = 1$, given by:*

$$\frac{\left\|ax-by\right\|-|a-b|}{\min\{a,b\}} \le \left\|x-y\right\| \le \frac{\left\|ax-by\right\|+|a-b|}{\max\{a,b\}}. \tag{11}$$

Inequality (11) represents a generalization of Maligranda's inequality, because, replacing x by $\frac{x}{\|x\|}$, y by $\frac{y}{\|y\|}$ and taking $a = \|x\|$, $b = \|y\|$, we deduce the inequalities from relation (3).

3. Some Inequalities Related to the Tapia Semi-Inner-Product

Next, we study estimates of the Cauchy–Schwarz inequality using the Tapia semi-inner-product. The Tapia semi-inner-product on the normed space X (see [17]) is the function $(\cdot,\cdot)_T : X \times X \to \mathbb{R}$, defined by

$$(x,y)_T := \lim_{\substack{t\to 0\\ t>0}} \frac{\varphi(x+ty)-\varphi(x)}{t},$$

where $\varphi(x) = \frac{1}{2}\|x\|^2,\ x \in X$.

The above limit exists for any pair of elements $x, y \in X$. The semi-inner-product have been used by many authors in several contexts (see, e.g., [18]). The Tapia semi-inner-product is positive homogeneous in each argument and satisfies inequality

$$|(x,y)_T| \le \|x\|\|y\|, \tag{12}$$

for all $x, y \in X$.

In the case when the norm $\|\cdot\|$ is generated by an inner product $\langle\cdot,\cdot\rangle$, then $(x,y)_T = \langle x,y\rangle$, for all $x, y \in X$, thus we find the Cauchy–Schwarz inequality.

Theorem 3. *If $X = (X, \|\cdot\|)$ is a normed vector space over the field of real numbers $\mathbb{R}$, then we have*

$$\|x\|\|y\| + (x,y)_T \le \left\|x\|y\| + y\|x\|\right\|. \tag{13}$$

for every vector x and y in X.

Proof. If $x = 0$ or $y = 0$, then the inequality of the statement is true. We consider $x \neq 0$ and $y \neq 0$. If in inequality (7) we replace a and b by $\|x\|$ and $t\|y\|$ with $t > 0$, then we obtain

$$\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, t\|y\|\} \leq \|x\| + t\|y\| - \|x + ty\|$$

$$\leq \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \max\{\|x\|, t\|y\|\},$$

which is equivalent to

$$\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \frac{1}{t} \min\{\|x\|, t\|y\|\} \leq \|y\| - \frac{\|x + ty\| - \|x\|}{t}$$

$$\leq \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \frac{1}{t} \max\{\|x\|, t\|y\|\}.$$

Thus, by passing to limit for $t \to 0, t > 0$, we deduce

$$\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} \min\{\|x\|, t\|y\|\} \leq \|y\| - \lim_{\substack{t \to 0 \\ t > 0}} \frac{\|x + ty\| - \|x\|}{t}$$

$$\leq \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} \max\{\|x\|, t\|y\|\}.$$

Because we have

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{\|x + ty\| - \|x\|}{t} = \lim_{\substack{t \to 0 \\ t > 0}} \frac{\|x + ty\|^2 - \|x\|^2}{t(\|x + ty\| + \|x\|)} = \frac{(x, y)_T}{\|x\|}$$

and for $t \to 0$, $t > 0$, we obtain $\min\{\|x\|, t\|y\|\} = t\|y\|$, equivalent to $\frac{1}{t} \min\{\|x\|, t\|y\|\} = \|y\|$. Then, we deduce the inequality

$$\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \|x\|\|y\| \leq \|x\|\|y\| - (x, y)_T, \tag{14}$$

which means that

$$\|x\|\|y\| + (x, y)_T \leq \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \|x\|\|y\| = \Big\| x\|y\| + y\|x\| \Big\|.$$

This inequality is equivalent with the inequality of the statement. □

Remark 3. *For nonzero elements $x, y \in X$, if we replace x by $\frac{x}{\|x\|}$ and y by $\frac{y}{\|y\|}$ in inequality (13), then we find the following inequality*

$$\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)_T \leq \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| - 1. \tag{15}$$

Let $X = (X, \langle \cdot, \cdot \rangle)$ be a vector space with inner product; then, for nonzero elements x, y, inequality (14) becomes

$$\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \|x\|\|y\| \leq \|x\|\|y\| - \langle x, y \rangle. \tag{16}$$

This inequality represents an improvement of Cauchy–Schwarz inequality.

For nonzero elements $x, y \in X$ denote $v(x,y) := \frac{x}{\|x\|} + \frac{y}{\|y\|}$, where X is a normed space. Then, inequality (14) becomes (see [15]):

$$(x,y)_T \le \|x\|\|y\|\big(\|v(x,y)\| - 1\big) \le \|x\|\|y\|, \tag{17}$$

for all nonzero vectors $x, y \in X$.

Theorem 4. *If $X = (X, \|\cdot\|)$ is a normed vector space over the field of real numbers $\mathbb{R}$, then we have*

$$0 \le \|x\|\big(\|x\| + \|y\| - \|x+y\|\big) \le \|x\|\|y\| - (x,y)_T \tag{18}$$

for every vector x and y in X.

Proof. Using the triangle inequality, we have $0 \le \|x\|\big(\|x\| + \|y\| - \|x+y\|\big)$. From relation (6), for $a = 1,\ b = t$, we deduce

$$\min\{1,t\}\big(\|x\| + \|y\| - \|x+y\|\big) \le \|x\| + t\|y\| - \|x+ty\|$$

$$\le \max\{1,t\}\big(\|x\| + \|y\| - \|x+y\|\big),$$

which divided by t becomes

$$\frac{\min\{1,t\}}{t}\big(\|x\| + \|y\| - \|x+y\|\big) \le \frac{\|x\| - \|x+ty\|}{t} + \|y\| \le \frac{\max\{1,t\}}{t}\big(\|x\| + \|y\| - \|x+y\|\big).$$

Thus, by passing to limit, we find the relation

$$\lim_{\substack{t\to 0\\ t>0}} \frac{\min\{1,t\}}{t}\big(\|x\| + \|y\| - \|x+y\|\big) \le \lim_{\substack{t\to 0\\ t>0}} \frac{\|x\| - \|x+ty\|}{t} + \|y\|.$$

Consequently, we obtain the inequality

$$\|x\| + \|y\| - \|x+y\| \le \|y\| - \frac{(x,y)_T}{\|x\|},$$

which implies the inequality of the statement. □

Remark 4. *(a) From inequality (18), we get:*

$$\|x\|^2 + (x,y)_T \le \|x\|\|x+y\|. \tag{19}$$

(b) If $X = (X, \langle\cdot,\cdot\rangle)$ is a vector space with inner product ($(x,y)_T = \langle x,y\rangle$), then inequality (18) is in fact a characterization between the triangle inequality and Cauchy–Schwarz inequality:

$$0 \le \|x\|\big(\|x\| + \|y\| - \|x+y\|\big) \le \|x\|\|y\| - \langle x,y\rangle, \tag{20}$$

for every vector x and y in X.

Relation (20) suggests the following:

Theorem 5. *If $X = (X, \langle\cdot,\cdot\rangle)$ is an inner product space over the field of complex numbers $\mathbb{C}$ and the norm $\|\cdot\|$ is generated by $\langle\cdot,\cdot\rangle$, then we have*

$$0 \le \|x\|\big(\|x\| + \|y\| - \|x+y\|\big) \le \|x\|\|y\| - Re\langle x,y\rangle \tag{21}$$

for every vector x and y in X.

Proof. Inequality (21) becomes

$$\|x\|\|x+y\| \geq \|x\|^2 + Re\langle x,y\rangle.$$

By squaring, we obtain

$$\|x\|^2\left(\|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2\right) \geq \|x\|^4 + 2\|x\|^2 Re\langle x,y\rangle + (Re\langle x,y\rangle)^2$$

which is equivalent to

$$\|x\|^2\|y\|^2 - (Re\langle x,y\rangle)^2 \geq 0,$$

which is true, for every vector x and y in X. □

Remark 5. *Since* $|\langle x,y\rangle| + Re\langle x,y\rangle \geq 0$*, from inequality (21), we deduce*

$$0 \leq \|x\|\big(\|x\| + \|y\| - \|x+y\|\big) \leq \|x\|\|y\| + |\langle x,y\rangle|, \tag{22}$$

for every vector x and y in X.

Below, we give a connection between the triangle inequality and the Cauchy–Schwarz inequality:

Theorem 6. *If $X = (X, \langle\cdot,\cdot\rangle)$ is an inner product space over the field of complex numbers $\mathbb{C}$ and the norm $\|\cdot\|$ is generated by $\langle\cdot,\cdot\rangle$, then we have*

$$\|x\| + \|y\| - \|x+y\| \geq \frac{\|x\|\|y\| - |\langle x,y\rangle|}{\|x\| + \|y\|} \tag{23}$$

for every vector x and y in X, $x \neq 0$ or $y \neq 0$.

Proof. From the equality

$$2\left(\|x\|\|y\| - Re\langle x,y\rangle\right) = 2\|x\|\|y\| - \left(\langle x,y\rangle + \overline{\langle x,y\rangle}\right) =$$

$$\|x\|^2 + 2\|x\|\|y\| + \|y\|^2 - \left(\|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2\right) = (\|x\| + \|y\|)^2 - \|x+y\|^2,$$

we deduce

$$\left(\|x\| + \|y\| + \|x+y\|\right)\left(\|x\| + \|y\| - \|x+y\|\right) \geq 2\left(\|x\|\|y\| - |\langle x,y\rangle|\right),$$

because $|\langle x,y\rangle| \geq Re\langle x,y\rangle$, which implies the following inequality

$$2\left(\|x\| + \|y\|\right)\left(\|x\| + \|y\| - \|x+y\|\right) \geq 2\left(\|x\|\|y\| - |\langle x,y\rangle|\right),$$

for every vector x and y in X. Consequently, we obtain the statement. □

4. Estimates for Numerical Radii via Cauchy–Schwarz and Triangle Inequalities

In this section, we employ the above results to obtain some new inequalities for the numerical radius and norm inequalities for Hilbert space operators.

Let $\mathbb{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. For $A \in \mathcal{B}(\mathcal{H})$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the operator norm of A, respectively. Recall that $\omega(A) = \sup\limits_{\|x\|=1} |\langle Ax,x\rangle|$. It is well-known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathbb{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{24}$$

In [19], Kittaneh gave the following estimate of the numerical radius which refines the first inequality in (24):

$$\frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\| \leq \omega^2 (A) . \tag{25}$$

For several other results of this kind, we refer the reader to papers [5,11,20–22].

We recall the following inequality for $\omega(A)$ which is known in the literature as the power inequality

$$\omega (A^n) \leq \omega^n (A) \tag{26}$$

for every $n \geq 1$. We denote by $\zeta_\lambda^A (a, b) = \inf_{\|x\|=1} \|(aA + b\lambda I)\, x\|^2$.

Theorem 7. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$(0 \leq)\, \|A\|^2 - \omega^2 (A) \leq \|A\| \, \|A + A^*\| . \tag{27}$$

Proof. From the inequality (3.11), we obtain

$$\|x\|^2 \leq \|x\| \|x + y\| + |\langle x, y\rangle| . \tag{28}$$

Putting $x = Ax$ and $y = A^* x$ with $\|x\| = 1$ in the inequality (28), we get

$$\|Ax\|^2 \leq \|Ax\| \, \|(A + A^*)\, x\| + \left|\left\langle A^2 x, x\right\rangle\right| . \tag{29}$$

Taking the supremum in (29) over $x \in \mathcal{H}$ with $\|x\| = 1$, we infer that

$$\|A\|^2 \leq \|A\| \, \|A + A^*\| + \omega\left(A^2\right) . \tag{30}$$

The inequality (27) follows by combining (26) and (30). □

Combining the second inequality in (24) and the inequality (30), we conclude the following result:

Theorem 8. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$(0 \leq)\, \omega^2 (A) - \omega\left(A^2\right) \leq \|A\| \, \|A + A^*\| . \tag{31}$$

Corollary 1. *Let $A \in \mathbb{B}(\mathcal{H})$ be an invertible operator and let $0 \neq \lambda \in \mathbb{C}$. Then,*

$$\left(\frac{\|A\| - \|A + \lambda I\|}{|\lambda|}\right) \|A\| \leq \omega (A) . \tag{32}$$

Proof. Replacing $x = Ax$ and $y = \lambda x$ with $\|x\| = 1$ and $0 \neq \lambda \in \mathbb{C}$ in the inequality (28), we obtain

$$\|Ax\|^2 \leq \|Ax\| \|(A + \lambda I)\, x\| + |\lambda| |\langle Ax, x\rangle| .$$

Now, taking supremum over unit vector $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|A\|^2 \leq \|A\| \|A + \lambda I\| + |\lambda| \omega (A) . \tag{33}$$

Now, if we divide (33) by $|\lambda| > 0$, then we get

$$\left(\frac{\|A\| - \|A + \lambda I\|}{|\lambda|}\right) \|A\| \leq \omega (A) .$$

This completes the proof. □

Theorem 9. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and let* $0 \neq \lambda \in \mathbb{C}$. *Then*

$$\|A\| - \omega(A) \leq \frac{(\|A\| + |\lambda|)^2 - \zeta_\lambda^A(1,1)}{|\lambda|}, \tag{34}$$

where $\zeta_\lambda^A(1,1) = \inf_{\|x\|=1} \|(A + \lambda I)x\|^2$.

Proof. It follows from Theorem 5 that

$$\|x\|\,\|y\| - |\langle x, y\rangle| \leq (\|x\| + \|y\|)^2 - (\|x\| + \|y\|)\,\|x + y\|\,.$$

From the triangle inequality, we infer that

$$\|x\|\,\|y\| - |\langle x, y\rangle| \leq (\|x\| + \|y\|)^2 - \|x + y\|^2.$$

Replacing $x = Ax$ and $y = \lambda x$ with $\|x\| = 1$ and $0 \neq \lambda \in \mathbb{C}$ in the above inequality, we get

$$\|Ax\|\,|\lambda| - |\langle Ax, x\rangle|\,|\lambda| \leq (\|Ax\| + |\lambda|)^2 - \|(A + \lambda I)x\|^2.$$

The above inequality implies,

$$\|Ax\| \leq \frac{(\|Ax\| + |\lambda|)^2 - \zeta_\lambda^A(1,1)}{|\lambda|} + |\langle Ax, x\rangle|\,.$$

Taking the supremum over all unit vectors $x \in \mathcal{H}$ gives

$$\|A\| \leq \frac{(\|A\| + |\lambda|)^2 - \zeta_\lambda^A(1,1)}{|\lambda|} + \omega(A)$$

as required. □

The following lemma contains a norm inequality for sums of positive operators that is sharper than the triangle inequality (see [23]).

Lemma 1. *Let* $A, B \in \mathbb{B}(\mathcal{H})$ *be two positive operators; then,*

$$\|A + B\| \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right). \tag{35}$$

The following theorem provides a refinement of the triangle inequality for general (i.e., not necessarily positive) operators.

Theorem 10. *Let* $A, B \in \mathbb{B}(\mathcal{H})$. *Then*

$$\|A + B\| \leq \sqrt{\frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \sqrt{\left(\|A\|^2 - \|B\|^2\right)^2 + 4\|AB^*\|^2}\right) + 2\omega(B^*A)}. \tag{36}$$

Proof. We have for any $x, y \in \mathcal{H}$,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x,y\rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2\,|\langle x,y\rangle|\end{aligned}$$

i.e.,

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\,|\langle x,y\rangle|\,. \tag{37}$$

Replacing x by Ax and y by Bx with $\|x\| = 1$ in the inequality (37), we infer that

$$\begin{aligned}\|(A+B)\,x\|^2 &\leq \|Ax\|^2 + \|Bx\|^2 + 2\,|\langle Ax, Bx\rangle| \\ &= \langle Ax, Ax\rangle + \langle Bx, Bx\rangle + 2\,|\langle B^*Ax, x\rangle| \\ &= \left\langle |A|^2 x, x\right\rangle + \left\langle |B|^2 x, x\right\rangle + 2\,|\langle B^*Ax, x\rangle| \\ &= \left\langle \left(|A|^2 + |B|^2\right) x, x\right\rangle + 2\,|\langle B^*Ax, x\rangle|\,.\end{aligned}$$

Thus,

$$\|(A+B)\,x\|^2 \leq \left\langle \left(|A|^2 + |B|^2\right) x, x\right\rangle + 2\,|\langle B^*Ax, x\rangle|\,.$$

By taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|A+B\|^2 \leq \left\||A|^2 + |B|^2\right\| + 2\omega\,(B^*A)\,. \tag{38}$$

On the other hand, from Lemma 1, replacing A with $|A|^2$ and B with $|B|^2$, we have

$$\left\||A|^2 + |B|^2\right\| \leq \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \sqrt{\left(\|A\|^2 - \|B\|^2\right)^2 + 4\|AB^*\|^2}\right), \tag{39}$$

where we use the following two identities

$$\||A|\,|B|\| = \|AB^*\|\,,$$

and

$$\left\||A|^2\right\| = \|A\|^2.$$

On making use of (38) and (39), we get

$$\|A+B\| \leq \sqrt{\frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \sqrt{\left(\|A\|^2 - \|B\|^2\right)^2 + 4\|AB^*\|^2}\right) + 2\omega\,(B^*A)}, \tag{40}$$

as required. □

Remark 6. *To show the inequality (40) improves the triangle inequality, we can write*

$$
\begin{aligned}
\|A+B\| &\leq \sqrt{\frac{1}{2}\left(\|A\|^2+\|B\|^2+\sqrt{\left(\|A\|^2-\|B\|^2\right)^2+4\|AB^*\|^2}\right)+2\omega\left(B^*A\right)} \\
&\leq \sqrt{\frac{1}{2}\left(\|A\|^2+\|B\|^2+\sqrt{\left(\|A\|^2-\|B\|^2\right)^2+4\|AB^*\|^2}\right)+2\left\|B^*A\right\|} \\
&\quad \text{(since } \omega\left(T\right)\leq\|T\|\text{, for any } T\in\mathbb{B}\left(\mathcal{H}\right)\text{)} \\
&\leq \sqrt{\frac{1}{2}\left(\|A\|^2+\|B\|^2+\sqrt{\left(\|A\|^2-\|B\|^2\right)^2+4\|A\|^2\|B\|^2}\right)+2\left\|A\right\|\left\|B\right\|} \\
&\quad \text{(by the submultiplicativity property of the operator norm)} \\
&= \sqrt{\|A\|^2+\|B\|^2+2\left\|A\right\|\left\|B\right\|} \\
&= \sqrt{\left(\|A\|+\|B\|\right)^2} \\
&= \|A\|+\|B\|\,.
\end{aligned}
$$

If, in the above inequalities, we take $B=A^*$, then we deduce the following:

Proposition 1. *Let $A\in\mathbb{B}\left(\mathcal{H}\right)$. Then,*

$$
\|A+A^*\|\leq\sqrt{\|A\|^2+\|A^2\|+2\omega\left(A^2\right)}. \tag{41}
$$

Remark 7. *Proposition 1 easily implies*

$$
\frac{1}{2}\left(\|A+A^*\|^2-\left(\|A\|^2+\left\|A^2\right\|\right)\right)\leq\omega\left(A^2\right). \tag{42}
$$

Now, on making use of the inequalities (42) and (26), we get

$$
\frac{1}{2}\left(\|A+A^*\|^2-\left(\|A\|^2+\left\|A^2\right\|\right)\right)\leq\omega^2\left(A\right). \tag{43}
$$

It is worth mentioning here that, if A is a self-adjoint operator, then (43) is a sharper inequality than (25).

Theorem 11. *Let $A\in\mathbb{B}\left(\mathcal{H}\right)$ and let $0\neq\lambda\in\mathbb{R}$. Then,*

$$
\min\{\frac{1}{c},\frac{1}{d}\}(c\|A\|+d|\lambda|-\|cA+d\lambda I\|)\leq\|A\|+|\lambda|-\zeta_\lambda^A(1,1), \tag{44}
$$

and

$$
\|A\|+|\lambda|-\|A+\lambda I\|\leq\max\{\frac{1}{c},\frac{1}{d}\}(c\|A\|+d|\lambda|-\zeta_\lambda^A(c,d)), \tag{45}
$$

where $\zeta_\lambda^A(c,d)=\inf_{\|x\|=1}\|(cA+d\lambda I)\,x\|^2$.

Proof. For $a=b$ in inequality (5), we have

$$
\begin{aligned}
\min\{\frac{1}{c},\frac{1}{d}\}(c\|x\|+d\|y\|-\|cx+dy\|)\leq\|x\|+\|y\|-\|x+y\|\leq \\
\leq\max\{\frac{1}{c},\frac{1}{d}\}(c\|x\|+d\|y\|-\|cx+dy\|),
\end{aligned} \tag{46}
$$

for all vectors x and y in X and $c, d \in \mathbb{R}_+^*$.

If we take the substitutions x by Ax and y by $\lambda x, \lambda \in \mathbb{R}$, with $\|x\| = 1$, then

$$\min\{\frac{1}{c}, \frac{1}{d}\}(c\|Ax\| + d|\lambda| - \|(cA + d\lambda I)x\|) \leq \|Ax\| + |\lambda| - \|(A + \lambda I)x\|\| \\ \leq \max\{\frac{1}{c}, \frac{1}{d}\}(c\|Ax\| + d|\lambda| - \|(cA + d\lambda I)x\|), \tag{47}$$

for all vectors x and y in X and $c, d \in \mathbb{R}_+^*$.

Taking the supremum in (47) over all unit vectors $x \in \mathcal{H}$ gives

$$\min\{\frac{1}{c}, \frac{1}{d}\}(c\|A\| + d|\lambda|) \leq \min\{\frac{1}{c}, \frac{1}{d}\}\|cA + d\lambda I\| + \|A\| + |\lambda| - \zeta_\lambda^A(1,1), \tag{48}$$

for all vectors x and y in X and $c, d \in \mathbb{R}_+^*$.

From inequality (46), we take the substitutions x by Ax and y by $\lambda x, \lambda \in \mathbb{R}$, with $\|x\| = 1$, then

$$\|Ax\| + |\lambda| \leq \|(A + \lambda I)x\| + \max\{\frac{1}{c}, \frac{1}{d}\}(c\|Ax\| + d|\lambda| - \|(cA + d\lambda I)x\|), \tag{49}$$

for all vectors x and y in X and $c, d \in \mathbb{R}_+^*$.

Taking the supremum over all unit vectors $x \in \mathcal{H}$, we deduce the statement. □

Remark 8. *If A is an invertible operator, then, for $\lambda = \omega(A)$ in inequalities (44) and (45), we have*

$$\min\{\frac{1}{c}, \frac{1}{d}\}(c\|A\| + d\omega(A) - \|cA + d\omega(A)I\|) \leq \|A\| + \omega(A) - \zeta_{\omega(A)}^A(1,1), \tag{50}$$

and

$$\|A\| + \omega(A) - \|A + \omega(A)I\| \leq \max\{\frac{1}{c}, \frac{1}{d}\}(c\|A\| + d\omega(A) - \zeta_{\omega(A)}^A(c,d)). \tag{51}$$

For $c = \omega(A)$ and $d = \|A\|$ in the above inequalities, we obtain

$$\|A\|\zeta_{\omega(A)}^A(1,1) - \|\omega(A)A + \|A\|\omega(A)I\| \leq \|A\|(\|A\| - \omega(A)), \tag{52}$$

and

$$\frac{1}{\omega(A)}(\zeta_{\omega(A)}^A(\omega(A), \|A\|) - \|A + \omega(A)I\| \leq \|A\|(\|A\| - \omega(A)). \tag{53}$$

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