Research Article

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An extension of Gronwall inequality in the theory of bodies with voids

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Abstract: In this paper, we obtain a generalization of the Gronwall's inequality to cover the study of porous elastic media considering their internal state variables. Based on some estimations obtained in three auxiliary results, we use this form of the Gronwall's inequality to prove the uniqueness of solution for the mixed initial-boundary value problem considered in this context. Thus, we can conclude that even if we take into account the internal variables, this fact does not affect the uniqueness result regarding the solution of the mixed initial-boundary value problem in this context.

Keywords: dipolar bodies, voids, internal state variables, Gronwall's inequality, uniqueness

1 Introduction

Researchers considered that the presence of internal state variables is necessary in a media to have a means to estimate mechanical properties of the respective bodies. As such, interest in these variables has grown rapidly in recent times. In addition, many authors consider that the theory of this kind of variables in different materials is represented by a material length scale and it is enough for a large number of applications in different theories of solid mechanics. It is considered that the order of the system's defining differential equation can suggest the minimum number of the internal state variables required to represent a given system. In the case of a system that is represented in transfer function form, it was established that the correct number of the internal state variables is like the order of the transfer function's denominator. However, we must not neglect that in the case that a reference state is transformed in a transfer form, it is possible to obtain a description specific to a stable system, and yet the original state can be unstable in some points. We can suggest, as important example, some electric circuits, in which the number of state variables can be the same as the number of energy storage elements in the circuit, such as capacitors and inductors.

It is known that the first results regarding the internal state variables appeared in the context of the theory of thermo-viscoelastic materials. We can suggest as an example the paper of Chirita [1]. Subsequently, these internal variables have been studied in the context of many other types of materials.

Therefore, Nachlinger and Nunziato approached in ref. [2] the internal state variables in the one-dimensional case of finite deformations without heat conduction.

The so-called Bammann internal state variable, approached in paper [3], has been highly successful in modeling some deformation processes in metals, which can be used with significant benefit to rocks, silicate, and other geological materials for modeling their dynamics of deformation. It can be said with certainty that the internal variables approach can offer some constitutive relations to take into account the change of history states because of the inelastic behavior of a poly-crystalline media. In the paper [4], the authors proposed a thermodynamically constituted framework to model a hysteresis of the capillarity in a saturated porous body, which is a dissipation mechanism, represented by a number of internal variables.

A constitutive model for amorphous thermoplastics using a thermodynamic approach with physically motivated internal state variables is formulated in the paper [5] and is inspired by current internal variable procedures that are specific for metals and are very different from those used to characterize the mechanical behavior of polymers.

In the paper [6], Anand and Gurtin formulated a theory for the elastic–viscoplastic deformation of amorphous

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bodies such as metallic and polymeric glasses. With the help of an internal-state variable that represents the local free-volume associated with certain metastable states, they can describe the highly nonlinear stress–strain behavior and give rise to post-yield strain softening. To model damage behavior, the authors of the study [7] proposed a representation of an internal variable based on the gradient theory. Their theory is based on some basic balance laws and suitable internal variables within a thermodynamic context.

We approached this topic in our study because it can be considered that the theory of media with internal state variables is part of the non-classical theories of continuous solids. Regarding the presence of the microstructure, which is specific to the works dedicated to generalized media, we must note that we followed the effect of voids and the fact that our bodies have a dipolar structure. Other aspects of the microstructure were addressed in studies [8–20].

Our study was developed according to the following plan. First we systematized the basic equations specific to the theory of porous media with internal state variables, such as the equations of motion, the equation of the equilibrated forces, the kinematic relations, and the constitutive equations. Then we set the boundary and initial conditions and we specified the necessary restrictions to obtain our results. In the first two theorems, we proved two auxiliary results, namely two conservation laws necessary for the main result. In the third theorem, we obtained a generalization of Gronwall inequality, adapted to our context. The main result is dedicated to the uniqueness of the solution of the mixed problem, this not being influenced by the presence of voids, of the dipolar structure of the internal state variables.

2 Basic equations

In our study, we will consider a domain D from R^3, the usual Euclidean space. At initial time $t = 0$, this domain is occupied by a porous elastic material with internal state variables.

The border of the region D will be denoted by ∂D, and we assume that the surface ∂D is smooth, closed, and bounded. Therefore, this enables application of the divergence theorem. The Cartesian vector and tensor notations are adopted. We will use the rule of summation regarding the repeated subscripts. A dot over a function f, for instance, is used to designate the partial derivative of the function f with regard to the time variable t, and a comma followed by a subscript will designate the partial differentiation of a function with respect to respective spatial argument.

The components of the normal to the boundary ∂D will be denoted by n_i. $\bar{D}$ represents the closure of D and we have $\bar{D} = D \cup \partial D$.

The behavior of a porous elastic dipolar body is described by the following specific variables:

$$v_k = v_k(t, x), \quad \phi_{kj} = \phi_{kj}(t, x),$$
$$\varphi = \varphi(t, x), \quad (t, x) \in [0, t_0) \times D,$$

where $v - (v_k)$ is the displacement vector field, $\phi - (\phi_{kj})$ is the dipolar displacement tensor field, and the scalar function φ is for characterizing material voids. It designates the change in volume fraction.

The internal state variables that can appear in the intimate structure of the body will be denoted by w_β, $\beta = 1, 2, \ldots, n$.

To characterize the strain measures, we use the tensors denoted by e_{mn}, ν_{mn}, and κ_{imn} defined by the following relations:

$$e_{mn} = \frac{1}{2}(v_{m,n} + v_{n,m}), \quad \nu_{mn} = v_{n,m} - \phi_{mn}, \quad \kappa_{imn} = \phi_{mn,i}, \tag{1}$$

which are called the strain–displacement relations (see, for instance, Eringen [21]).

In the theory of porous dipolar bodies, three tensors are defined, such as t_{mn}, σ_{mn}, and η_{mni}, which are intrinsically linked to the above strain tensors by means of the following constitutive relations:

$$\begin{aligned} t_{mn} &= A_{mnij} e_{ij} + G_{ijmn} \nu_{mn} + F_{ijmnr} \kappa_{ijr} + a_{mn} \varphi \\ &\quad + A_{mnk} \varphi_{,k} + P_{mn\beta} w_\beta, \\ \sigma_{mn} &= G_{ijmn} e_{ij} + B_{mnij} \nu_{ij} + D_{mnijr} \kappa_{ijr} + b_{mn} \varphi \\ &\quad + B_{mnk} \varphi_{,k} + Q_{mn\beta} w_\beta, \\ \eta_{kmn} &= F_{ijkmn} e_{ij} + D_{kmnij} \nu_{ij} + C_{kmnijk} \kappa_{ijk} \\ &\quad + c_{kmn} \varphi + C_{kmni} \varphi_{,i} + R_{kmn\beta} w_\beta. \end{aligned} \tag{2}$$

The components of the equilibrated stress vector g_k and the intrinsic equilibrated force h will be determined by means of the following two constitutive relations:

$$\begin{aligned} g_k &= A_{mnk} e_{mn} + B_{mnk} \nu_{mn} + C_{mnik} \kappa_{mni} + d_k \varphi + g_{mk} \varphi_{,m}, \\ h &= -a_{mn} e_{mn} - b_{mn} \nu_{mn} - c_{mni} \kappa_{mni} - \xi \varphi - d_k \varphi_{,k}. \end{aligned} \tag{3}$$

According to Eringen [21], the basic system of differential equations, in the context of theory of porous elastic bodies having internal variables, is made of:

– the motion equations:

$$\begin{aligned} (t_{mn} + \sigma_{mn})_{,n} + \varrho F_m &= \varrho \ddot{v}_m, \\ \eta_{mnk,m} + \sigma_{nk} + \varrho G_{nk} &= I_{nr} \ddot{\phi}_{kr}; \end{aligned} \tag{4}$$

– the equation of the equilibrated forces:

$$g_{k,k} + h + \varrho l = \varrho \ddot{\varphi}. \tag{5}$$

Using a suggestion given by Chirita in ref. [1], it can be shown that within a linear approximation, the internal variables satisfy the equations:

$$\dot{w}_\beta = L_\beta, \quad \beta = 1, 2, \ldots, n, \tag{6}$$

where the L_β functions are determined with the help of the following constitutive equations:

$$L_\beta = p_{mn\beta} e_{mn} + q_{mn\beta} \nu_{mn} + r_{mnk\beta} \kappa_{mnk} + d_\beta \varphi + h_{\alpha\beta} w_\alpha. \tag{7}$$

We have the following meanings for the notations used in the above equations:

ϱ – the constant mass density;
t_{mn}, σ_{mn}, and η_{mnk} – the stress tensors;
I_{mn} – the tensor of inertia;
F_m – the body force;
G_{mn} – the dipolar body force;
e_{mn}, ν_{mn}, and κ_{mnk} – the tensors of strain.

In addition, coefficients $A_{mnij}, B_{mnij}, \ldots, a_{ijk}, \ldots, h_{\alpha\beta}$, used in the above relations, are functions depending on points $x = (x_m)$ of the body and characterize the elastic characteristics of a body having some internal state variables (called, also, the constitutive coefficients). In the case that the porous elastic body is a homogeneous one, then the above-defined characteristics are constants. It is assumed the above constitutive characteristics satisfy some symmetry relations of the form:

$$\begin{aligned} &A_{mnij} = A_{ijmn} = A_{mnji}, \quad B_{mnij} = B_{ijmn}, \\ &G_{mnij} = G_{nmji}, \quad F_{mnkij} = F_{mnkji}, \quad A_{mnrijk} = A_{ijkmnr}, \\ &a_{mn} = a_{nm}, \quad A_{mnk} = A_{nmk}, \quad P_{mn\beta} = P_{nm\beta}, \\ &p_{mn\beta} = p_{nm\beta}. \end{aligned} \tag{8}$$

To constitute the mixed problem in this context, we must indicate the initial conditions:

$$\begin{aligned} &v_m(0, x) = v_m^0(x), \quad \dot{v}_m(0, x) = v_m^1(x), \\ &\phi_{mn}(0, x) = \phi_{mn}^0(x), \quad \dot{\phi}_{mn}(0, x) = \phi_{mn}^1(x), \\ &\varphi(0, x) = \varphi^0(x), \quad w_\beta(0, x) = w_\beta^0(x), \quad (x) \in D, \end{aligned} \tag{9}$$

and the following boundary relations:

$$\begin{aligned} &v_m = \tilde{v}_m, \text{ on } [0, t_0] \times \overline{\partial D_1}, \ (t_{kl} + \sigma_{kl}) n_k = \tilde{t}_l, \\ &\text{on } [0, t_0] \times \partial D_2, \\ &\phi_{kl} = \tilde{\phi}_{kl}, \text{ on } [0, t_0] \times \overline{\partial D_3}, \ \eta_{klm} n_k = \tilde{\eta}_{lm}, \\ &\text{on } [0, t_0] \times \partial D_4, \\ &\varphi = \tilde{\varphi}, \text{ on } [0, t_0] \times \overline{\partial D_5}, \ g_k n_k = \tilde{g}, \\ &\text{on } [0, t_0] \times \partial D_6, \end{aligned} \tag{10}$$

where $\overline{\partial B_1}$, ∂B_2, $\overline{\partial B_3}$, ∂B_4, $\overline{\partial B_5}$, and ∂B_6 are subsets of the surface ∂B, defined such that:

$$\begin{aligned} &\overline{\partial B_1} \cup \partial B_2 = \overline{\partial B_3} \cup \partial B_4 = \overline{\partial B_5} \cup \partial B_6 = \partial B, \\ &\partial B_1 \cap \partial B_2 = \partial B_3 \cap \partial B_4 = \partial B_5 \cap \partial B_6 = \varnothing. \end{aligned}$$

The functions v_m^0, v_m^1, ϕ_{mn}^0, ϕ_{mn}^1, φ^0, w_β^0, $\tilde{v}_m$, $\tilde{t}_m$, $\tilde{\phi}_{mn}$, $\tilde{\eta}_{jk}$, $\tilde{\varphi}$, and $\tilde{g}$, which appear in the relations (9) and (10), are known functions, at all points where they are defined.

We will denote by $\mathcal{P}$ the boundary-initial value problem from the theory of porous dipolar bodies having internal state variables. It includes the basic equations (4)–(6), the boundary relations from (10), and the initial data from (9).

We will call the solution to the problem $\mathcal{P}$ a state of deformation $(v_m, \phi_{mn}, \varphi, w_\beta)$, which satisfies equations (4)–(6) and the conditions (9) and (10).

3 Basic results

At the beginning of this section, we will systematize the assumptions that must be satisfied to obtain our results. Then we will prove a functional equality and two integral inequalities, which will be essential for proving the main result of our study, namely the theorem on the uniqueness of the solution of the problem $\mathcal{P}$.

Therefore, we will use the next hypotheses:

(i) the density ϱ is considered strictly positive, i.e.,

$$\varrho(x) \geq \varrho_0 > 0, \text{ on } D;$$

(ii) there is the constant $c_1 > 0$, so that:

$$I_{mn} \omega_m \omega_n \geq \lambda_1 \omega_m \omega_m, \ \forall \omega_m;$$

(iii) the constitutive tensors A_{mnkl}, B_{mnkl}, and C_{mnrjkl} are positive definite:

$$\begin{aligned} &\int_D A_{mnkl} \omega_{kl} \omega_{mn} \mathrm{d}V \geq c_2 \int_D \omega_{mn} \omega_{mn} \mathrm{d}V, \quad \forall \omega_{mn}, \\ &\int_D B_{mnkl} \omega_{kl} \omega_{mn} \mathrm{d}V \geq c_3 \int_D \omega_{mn} \omega_{mn} \mathrm{d}V, \quad \forall \omega_{mn}, \\ &\int_D C_{mnrjkl} \omega_{jkl} \omega_{mnr} \mathrm{d}V \geq c_4 \int_D \omega_{mnr} \omega_{mnr} \mathrm{d}V, \quad \forall \omega_{mnr}, \end{aligned}$$

where c_2, c_3, and c_4 are positive constants.

The procedure used for the uniqueness of the solution is a common one.

We assume that the problem $\mathcal{P}$ has two solutions, and we prove that their difference is zero.

The mathematical apparatus used for the demonstration is different, namely Gronwall's inequality, adapted to the mentioned context.

Let us consider two different solutions of the mixed problem $\mathcal{P}$:

$$(v_m^{(\alpha)}, \phi_{mn}^{(\alpha)}, \varphi^{(\alpha)}, w_\beta^{(\alpha)}), \quad \alpha = 1, 2.$$

If we take into account that the mixed problem $\mathcal{P}$ is linear, we deduce that the difference of its two solutions is also a solution.

Let us denote by $(u_m, \psi_{mn}, \kappa, \omega_\beta)$ the differences, i.e.,

$$u_m = v_m^{(2)} - v_m^{(1)}, \quad \psi_{mn} = \phi_{mn}^{(2)} - \phi_{mn}^{(1)},$$
$$\kappa = \varphi^{(2)} - \varphi^{(1)}, \quad \omega_\beta = w_\beta^{(2)} - w_\beta^{(1)}.$$

Clearly, the state of deformation $(u_m, \psi_{mn}, \kappa, \omega_\beta)$ is a solution of the problem $\mathcal{P}$, that is, it satisfies the equations (4)–(6), in the particular case:

$$F_m = G_{mn} = l = 0,$$

and the conditions (9) and (10), in which

$$v_m^0 = v_m^1 = \phi_{mn}^0 = \phi_{mn}^1 = \varphi^0 = w_\beta^0,$$

and

$$\tilde{v}_m = \tilde{t}_m \tilde{\phi}_{mn} = \tilde{\eta}_{mn} = \tilde{\varphi} = \tilde{g} = 0.$$

We will show that, based on hypotheses (i)–(iii), the above considerations lead to the conclusion that:

$$u_m = \psi_{mn} = \kappa = \omega_\beta = 0,$$

in $[0, t_0] \times D$.

To simplify the writing, instead of the problem $\mathcal{P}$ we will consider a simpler problem, which we denote by $\mathcal{P}_0$.

For the problem $\mathcal{P}_0$, the loads are missing in the basic equations, so they have the form:

$$\begin{aligned}
&(t_{mn} + \sigma_{mn})_{,n} = \varrho \ddot{v}_m,\\
&\eta_{kmn,k} + \sigma_{mn} = I_{mr} \ddot{\phi}_{nr},\\
&g_{k,k} + h = \varrho \ddot{\varphi}.
\end{aligned} \tag{11}$$

Together with equation (11), we must consider the differential equations of the internal state variables:

$$\dot{w}_\beta = L_\beta. \tag{12}$$

Moreover, in problem $\mathcal{P}_0$, both the initial conditions and the boundary conditions appear in their homogeneous form, i.e.,

$$\begin{aligned}
&u_i(x_s, 0) = 0, \quad \dot{u}_i(x_s, 0) = 0, \quad \varphi_{ij}(x_s, 0) = 0,\\
&\dot{\varphi}_{ij}(x_s, 0) = 0, \quad \theta(x_s, 0) = 0, \quad w_\beta(x_s, 0) = 0,\\
&(x_s) \in D
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&u_i = 0, \ \text{on } \overline{\partial B_1} \times [0, t_0],\\
&t_i \equiv (\tau_{ij} + \sigma_{ij}) n_j = 0, \ \text{on } \partial B_2 \times [0, t_0],\\
&\varphi_{ij} = 0, \ \text{on } \overline{\partial B_3} \times [0, t_0],\\
&\eta_{jk} \equiv \eta_{ijk} n_i = 0, \ \text{on } \partial B_4 \times [0, t_0],\\
&\theta = 0, \ \text{on } \overline{\partial B_5} \times [0, t_0], \quad q \equiv q_i n_i = 0,\\
&\text{on } \partial B_6 \times [0, t_0].
\end{aligned} \tag{14}$$

In the following, together with the problem $\mathcal{P}_0$, we will consider the constitutive relations (2), (3), and (7).

In the following three theorems we will prove some estimates, as auxiliary results, necessary to prove the uniqueness result.

Theorem 1. *If $(v_m, \phi_{mn}, \varphi, w_\beta)$ is a solution of the problem $\mathcal{P}_0$, then we can find a constant $M > 0$ that is part of the following inequality:*

$$\begin{aligned}
&\int_D (P_{mn\beta} e_{mn} + Q_{mn\beta} v_{mn} + R_{jmn\beta} \kappa_{jmn}) \dot{w}_\beta \, dV\\
&\quad \le M \int_B (e_{mn} e_{mn} + v_{mn} v_{mn} + \kappa_{jmn} \kappa_{jmn} + \varphi^2\\
&\qquad + w_\beta w_\beta) \, dV,
\end{aligned} \tag{15}$$

for any $t \in [0, t_0]$.

Proof. Using the relations (6) and (7), we can write:

$$\begin{aligned}
&\int_D (P_{mn\beta} e_{mn} + Q_{mn\beta} v_{mn} + R_{kmn\beta} \kappa_{jmn}) \dot{w}_\beta \, dV\\
&= \int_D (P_{mn\beta} e_{mn} + Q_{mn\beta} v_{mn} + R_{jmn\beta} \kappa_{jmn})(p_{mn\beta} e_{mn}\\
&\quad + q_{mn\beta} v_{mn} + r_{mnk\beta} \kappa_{mnk} + d_\beta \varphi + h_{\alpha\beta} w_\alpha) \, dV\\
&= \int_B (\bar{A}_{ijmn} e_{ij} e_{mn} + \bar{D}_{ijmn} e_{ij} v_{mn} + \bar{F}_{ijmnl} e_{ij} \kappa_{mnl}\\
&\quad + \bar{B}_{ijmn} v_{ij} v_{mn} + \bar{G}_{ijmnl} v_{ij} \kappa_{mnl} + \bar{C}_{ijsmnr} \kappa_{ijs} \kappa_{mnr}\\
&\quad + \bar{A}_{mn} e_{mn} \varphi + \bar{A}_{mn\alpha} e_{mn} w_\alpha + \bar{B}_{mn} v_{mn} \varphi\\
&\quad + \bar{B}_{mn\alpha} v_{mn} w_\alpha + \bar{C}_{mnj} \kappa_{mnj} \varphi + \bar{C}_{mnj\alpha} \kappa_{mnj} w_\alpha) \, dV.
\end{aligned} \tag{16}$$

The notations used in the above identity are as follows:

$$\begin{aligned}
&\bar{A}_{ijmn} = \frac{1}{2}(P_{mn\beta} p_{ij\beta} + P_{ij\beta} p_{mn\beta}),\\
&\bar{D}_{ijmn} = P_{mn\beta} q_{ij\beta} + Q_{ij\beta} p_{mn\beta},\\
&\bar{F}_{ijmnk} = P_{ij\beta} r_{mnk\beta} + R_{mnk\beta} p_{ij\beta},\\
&\bar{B}_{ijmn} = \frac{1}{2}(Q_{ij\beta} q_{mn\beta} + Q_{mn\beta} q_{ij\beta}),\\
&\bar{G}_{ijmnk} = Q_{ij\beta} r_{mnk\beta} + R_{mnk\beta} q_{ij\beta},\\
&\bar{C}_{ijkmnl} = \frac{1}{2}(R_{ijk\beta} r_{mnl\beta} + R_{mnl\beta} r_{ijk\beta}),\\
&\bar{A}_{mn} = P_{mn\beta} d_\beta, \quad \bar{A}_{mn\alpha} = P_{mn\beta} h_{\alpha\beta},\\
&\bar{B}_{mn} = Q_{mn\beta} d_\beta, \quad \bar{B}_{mn\alpha} = Q_{mn\beta} h_{\alpha\beta},\\
&\bar{C}_{mnj} = R_{mnj\beta} d_\beta, \quad \bar{C}_{mnj\alpha} = R_{mnj\beta} h_{\alpha\beta}.
\end{aligned} \tag{17}$$

For each term in the last integral from identity (16), we will apply the Schwarz's inequality and a geometric-arithmetic mean inequality of the form:

$$xy \le \frac{1}{2}\left(\frac{x^2}{\delta^2} + y^2\delta^2\right), \tag{18}$$

to obtain the inequality from Theorem 1.3.

To this aim we introduce the notations:

$$\begin{aligned}
M_1^2 &= 2\max(\bar{A}_{ijmn}\bar{A}_{ijmn})(x),\\
M_2^2 &= \max(\bar{D}_{ijmn}\bar{D}_{ijmn})(x),\\
M_3^2 &= \max(\bar{F}_{ijmnl}\bar{F}_{ijmnl})(x),\\
M_4^2 &= 2\max(\bar{B}_{ijmn}\bar{B}_{ijmn})(x),\\
M_5^2 &= \max(\bar{G}_{ijmnl}\bar{G}_{ijmnl})(x),\\
M_6^2 &= 2\max(\bar{C}_{ijsmnr}\bar{C}_{ijsmnr})(x),\\
M_7^2 &= \max(\bar{A}_{mn}\bar{A}_{mn})(x),\\
M_8^2 &= \max(\bar{A}_{mn\alpha}\bar{A}_{mn\alpha})(x),\\
M_9^2 &= \max(\bar{B}_{mn}\bar{B}_{mn})(x),\\
M_{10}^2 &= \max(\bar{B}_{mn\alpha}\bar{B}_{mn\alpha})(x),\\
M_{11}^2 &= \max(\bar{C}_{mnj}\bar{C}_{mnj})(x),\\
M_{12}^2 &= \max(\bar{C}_{mnj\alpha}\bar{C}_{mnj\alpha})(x).
\end{aligned} \tag{19}$$

With these estimates, we can find the arbitrary positive constants $\delta_1, \delta_2, \ldots, \delta_{12}$, such that from (16) we deduce:

$$\begin{aligned}
&\frac{1}{2}\int_D (P_{mn\beta}e_{mn} + Q_{mn\beta}\nu_{mn} + R_{kmn\beta}\kappa_{jmn})\dot{w}_\beta \mathrm{d}V\\
&\le \left(M_1^2 + \frac{M_2^2}{\delta_2^2} + \frac{M_3^2}{\delta_3^2} + \frac{M_7^2}{\delta_7^2} + \frac{M_8^2}{\delta_8^2}\right)\int_D e_{mn}e_{mn}\mathrm{d}V\\
&+ \left(M_2^2\delta_2^2 + M_4^2 + \frac{M_5^2}{\delta_5^2} + \frac{M_9^2}{\delta_9^2} + \frac{M_{10}^2}{\delta_{10}^2}\right)\int_B \nu_{ij}\nu_{ij}\mathrm{d}V\\
&+ \left(M_6^2 + M_3^2\delta_3^2 + M_5^2\delta_5^2 + \frac{M_{11}^2}{\delta_{11}^2} + \frac{M_{12}^2}{\delta_{12}^2}\right)\int_B \kappa_{ijs}\kappa_{ijs}\mathrm{d}V\\
&+ (M_7^2\delta_7^2 + M_9^2\delta_9^2 + M_{11}^2\delta_{11}^2)\int_B \varphi^2\mathrm{d}V\\
&+ (M_8^2\delta_8^2 + M_{10}^2\delta_{10}^2 + M_{12}^2\delta_{12}^2)\int_B w_\beta w_\beta\mathrm{d}V.
\end{aligned} \tag{20}$$

Finally, it is sufficient to choose the positive constant M, so that:

$$\begin{aligned}
M = \frac{1}{2}\max\Big\{&M_1^2 + \frac{M_2^2}{\delta_2^2} + \frac{M_3^2}{\delta_3^2} + \frac{M_7^2}{\delta_7^2} + \frac{M_8^2}{\delta_8^2},\\
&M_2^2\delta_2^2 + M_4^2 + \frac{M_5^2}{\delta_5^2} + \frac{M_9^2}{\delta_9^2} + \frac{M_{10}^2}{\delta_{10}^2},\\
&M_6^2 + M_3^2\delta_3^2 + M_5^2\delta_5^2 + \frac{M_{11}^2}{\delta_{11}^2} + \frac{M_{12}^2}{\delta_{12}^2},\\
&M_7^2\delta_7^2 + M_9^2\delta_9^2 + M_{11}^2\delta_{11}^2,\\
&M_8^2\delta_8^2 + M_{10}^2\delta_{10}^2 + M_{12}^2\delta_{12}^2\Big\}.
\end{aligned} \tag{21}$$

It is easy to see that if we take into account (21), from (20) we get the estimate (15), which ends the proof of Theorem 1. □

Theorem 2. *Consider the ordered array* $(\nu_m, \phi_{mn}, \varphi, w_\beta)$ *that satisfies the problem* $\mathcal{P}_0$. *Then we have the following identity*:

$$\begin{aligned}
&\frac{1}{2}\int_D (A_{mnkl}e_{kl}e_{mn} + 2G_{klmn}e_{kl}\nu_{mn} + 2F_{mnrkl}e_{kl}\kappa_{mnr}\\
&+ B_{klmn}\nu_{kl}\nu_{mn} + A_{jlsmnr}\kappa_{jls}\kappa_{mnr} + 2D_{klmnr}\nu_{kl}\kappa_{mnr}\\
&+ 2a_{mn}e_{mn}\varphi + 2P_{mn\beta}e_{mn}w_\beta + 2b_{mn}\nu_{mn}\varphi\\
&+ 2Q_{mn\beta}\nu_{mn}w_\beta + 2c_{mnr}\kappa_{mnr}\varphi + 2R_{mnr\beta}\kappa_{mnr}w_\beta\\
&+ \varrho\dot{\nu}_m\dot{\nu}_m + I_{kr}\dot{\phi}_{jr}\dot{\phi}_{jk} + \varrho\dot{\varphi}^2)\mathrm{d}V\\
&= \int_0^t\int_D (P_{mn\beta}e_{mn} + Q_{mn\beta}\nu_{mn} + R_{kmn\beta}\kappa_{jmn})\dot{w}_\beta \mathrm{d}V\mathrm{d}s.
\end{aligned} \tag{22}$$

Proof. Taking into account the symmetry relations (8), with the help of the constitutive equation (2) we are led to the following estimate:

$$\begin{aligned}
&t_{mn}\dot{e}_{mn} + \sigma_{mn}\dot{\nu}_{mn} + \eta_{mnr}\dot{\kappa}_{mnr}\\
&= \frac{1}{2}\frac{\partial}{\partial t}(A_{mnkl}e_{kl}e_{mn} + 2G_{mnkl}e_{kl}\nu_{mn} + 2F_{mnrkl}e_{kl}\kappa_{mnr}\\
&+ B_{mnkl}\nu_{kl}\nu_{mn} + A_{ijsmnr}\kappa_{ijs}\kappa_{mnr} + 2D_{klmnr}\nu_{kl}\kappa_{mnr}\\
&+ 2a_{mn}e_{mn}\varphi + 2P_{mn\beta}e_{mn}w_\beta + 2b_{mn}\nu_{mn}\varphi\\
&+ 2Q_{mn\beta}\nu_{mn}w_\beta + 2c_{mnr}\kappa_{mnr}\varphi + 2R_{mnr\beta}\kappa_{mnr}w_\beta)\\
&- (P_{mn\beta}e_{mn} + Q_{mn\beta}\nu_{mn} + R_{kmn\beta}\kappa_{jmn})\dot{w}_\beta.
\end{aligned} \tag{23}$$

For the right-hand limb of equality (23), we can obtain an equivalent formulation, if we use the equations of motion (4) and the equation of the equilibrated forces (5):

$$\begin{aligned}
&t_{mn}\dot{e}_{mn} + \sigma_{mn}\dot{\nu}_{mn} + \eta_{mnr}\dot{\kappa}_{mnr}\\
&= [(t_{mn} + \sigma_{mn})\dot{\nu}_m + \eta_{kmn}\dot{\phi}_{km} + g_n\varphi]_{,n}\\
&- \frac{1}{2}\frac{\partial}{\partial t}(\varrho\dot{\nu}_m\dot{\nu}_m + I_{kr}\dot{\phi}_{jr}\dot{\phi}_{jk} + \varrho\dot{\varphi}^2).
\end{aligned} \tag{24}$$

If we equalize the right members of equalities (23) and (24), it will lead to the equality:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(&A_{mnkl}e_{kl}e_{mn}+2G_{mnkl}e_{kl}v_{mn}+2F_{mnrkl}e_{kl}\kappa_{mnr}\\&+B_{mnkl}v_{kl}v_{mn}+A_{ijsmnr}\kappa_{ijs}\kappa_{mnr}+2D_{klmnr}v_{kl}\kappa_{mnr}\\&+2a_{mn}e_{mn}\varphi+2P_{mn\beta}e_{mn}w_\beta+2b_{mn}v_{mn}\varphi\\&+2Q_{mn\beta}v_{mn}w_\beta+2c_{mnr}\kappa_{mnr}\varphi+2R_{mnr\beta}\kappa_{mnr}w_\beta\\&+\varrho\dot{v}_m\dot{v}_m+I_{kr}\dot{\phi}_{jr}\dot{\phi}_{jk}+\varrho\dot{\varphi}^2)\\&=[(t_{mn}+\sigma_{mn})\dot{v}_m+\eta_{kmn}\dot{\phi}_{km}+g_n\varphi]_{,n}\\&\quad+(P_{mn\beta}e_{mn}+Q_{mn\beta}v_{mn}+R_{jmn\beta}\kappa_{jmn})\dot{w}_\beta.\end{aligned}\tag{25}$$

After integrating the equality (25) on volume D, we apply the divergence theorem and take into account the homogeneous boundary relations (14), which will lead to the next identity:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_D(&A_{mnkl}e_{kl}e_{mn}+2G_{mnkl}e_{kl}v_{mn}+2F_{mnrkl}e_{kl}\kappa_{mnr}\\&+B_{mnkl}v_{kl}v_{mn}+A_{ijsmnr}\kappa_{ijs}\kappa_{mnr}+2D_{klmnr}v_{kl}\kappa_{mnr}\\&+2a_{mn}e_{mn}\varphi+2P_{mn\beta}e_{mn}w_\beta+2b_{mn}v_{mn}\varphi\\&+2Q_{mn\beta}v_{mn}w_\beta+2c_{mnr}\kappa_{mnr}\varphi+2R_{mnr\beta}\kappa_{mnr}w_\beta\\&+\varrho\dot{v}_m\dot{v}_m+I_{kr}\dot{\phi}_{jr}\dot{\phi}_{jk}+\varrho\dot{\varphi}^2)\mathrm{d}V\\&=+\int_D(P_{mn\beta}e_{mn}+Q_{mn\beta}v_{mn}+R_{kmn\beta}\kappa_{jmn})\dot{w}_\beta\mathrm{d}V.\end{aligned}\tag{26}$$

Now it is easy to obtain the equality (22) from the statement of Theorem 2, because we only have to integrate the equality (26) on the interval $[0, t]$ and take into account the initial data in their homogeneous form from (13). The proof of Theorem 2 is now completed. □

Using a procedure very similar to the one used in the proof of Theorem 1, we can prove the following theorem, the last of the auxiliary theorems.

Theorem 3. *Let* $(v_m, \phi_{mn}, \varphi, w_\beta)$ *be a solution of the mixed homogeneous problem* $\mathcal{P}_0$. *Then we can find a positive constant C, and so it satisfies the next inequality.*

$$\begin{aligned}\int_D&(\dot{v}_m\dot{v}_m+\dot{\phi}_{mn}\dot{\phi}_{mn}+e_{mn}e_{mn}+v_{mn}v_{mn}\\&+\kappa_{mnr}\kappa_{mnr}+\varphi^2+w_\beta w_\beta)\mathrm{d}V\\&\le C\int_0^t\int_D(\dot{v}_m\dot{v}_m+\dot{\phi}_{mn}\dot{\phi}_{mn}+e_{mn}e_{mn},\\&+v_{mn}v_{mn}+\kappa_{mnr}\kappa_{mnr}+\varphi^2+w_\beta w_\beta)\mathrm{d}V\mathrm{d}s,\end{aligned}\tag{27}$$

for any $t\in[0, t_0]$.

Estimates obtained using the inequality of Schwarz and a geometric–arithmetic mean inequality in the form from (18) are used for the proof.

Remark. It is easy to see that inequality (27) is a generalization of Gronwall's known inequality.

Based on the auxiliary results from Theorems 1–3, we can now approach the basic result of our study, namely we will prove the uniqueness of solution of mixed problem in the theory of elastic porous body considering its internal variables.

Theorem 4. *If we suppose that the hypotheses (i)–(iii) hold, then the mixed problem* $\mathcal{P}$*, from the theory of porous elastic media having internal variables, admits at most one solution.*

Proof. Suppose, by reduction to the absurd, that the problem $\mathcal{P}$ admits two different solutions. Because of the linearity of the $\mathcal{P}$ problem, the difference between the two solutions is also a solution, but for the $\mathcal{P}_0$ problem, previously defined. Our intention is to prove that the difference between the two solutions is zero. To this aim, we define the quadratic form $d(t)$ by:

$$\begin{aligned}d(t)\ =\ \int_B&(\dot{u}_i\dot{u}_i+\dot{\varphi}_{ij}\dot{\varphi}_{ij}+e_{ij}e_{ij}+v_{ij}v_{ij}\\&+\kappa_{ijr}\kappa_{ijr}+\theta^2+w_\beta w_\beta)\mathrm{d}V,\end{aligned}$$

and we will prove that $d(t)=0,\ \forall t\in[0, t_0]$.

Because the difference of the solutions is a solution of the problem $\mathcal{P}_0$, then with the help of Gronwall's inequality (27) we deduce $d(t)\le 0$, which is absurdum, the integrant being a positive function. Therefore, we deduce:

$$\dot{v}_m=0,\quad \dot{\phi}_{mn}=0,\quad \varphi=0,\quad w_\beta=0.$$

If we take into account that the data on the boundary are zero, we obtained that the difference between the two solutions is zero on $[0, t_0]$, and the proof of Theorem 4 is complete. □

4 Conclusion

With the help of three auxiliary results (two identities and an inequality) we obtain a generalization of Gronwall inequality, in our context of the theory of porous media with internal state variables. This inequality has helped us to demonstrate that the solution to the mixed problem in this context is unique. Therefore, we can deduce that even if the voids are taken into account, the dipolar structure and the internal variables, these do not change the uniqueness result of the solution of the considered problem.

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