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New analytical formalisms used in finite element analysis of robots with elastic elements

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ABSTRACT

Obtaining the equations of motion for an element in finite element analysis (FEA) model in the analysis of a multi-body system (MBS) having component elastic elements represents an important (maybe the main) step to build a software able to solve such a problem numerically. In use FEA in the study of a MBS with elastic elements, the method of Lagrange's equations is especially used at present. This method presents the advantages of a homogeneous writing and the possibility to follow the operations easier. However, there are also equivalent formulations, developed by analytical mechanics, for approaching such a mechanical system. The earning of these alternative forms will be presented, by comparison.

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1. Introduction

The analysis of complex mechanical systems uses, mainly, methods of analytical mechanics that represent the best way of approaching such systems. Obviously, there are a lot of equivalent approaches in mechanics that allow obtaining the motion equations of a constraint mechanical system, but in the case of analyzing some elastic systems with the finite element method (FEM), the analytical approaches become the most suitable for studying such problems. The main reason is the complexity of the analyzed systems and the possibility of using a formalism that allows the analysis of a system with a great DOF. To study such systems it is no longer necessary to study each element of the system, but the equations written for a single type of finite element remain valid for all identical finite elements which discretizes the structure. Analytical mechanics represents a form of approach to classical mechanics, using the basic axioms of classical mechanics, through a collection of alternative mathematical formulations, related to each other and equivalent to obtain the final results. If Newtonian mechanics mainly use vectors, analytical mechanics generalizes the presentation mode using kinematic and dynamic scalar elements of motion that define the system. The constraints (due to the liaisons) that exist in a system reduce the number of DOF and the analytical mechanics is the most appropriate framework to describe the effect of these constraints. The coordinates that characterize the movement can be of any type, which is why they are called generalized

coordinates. The basic notions in the analytical descriptions are represented by the kinetic energy, the work of the external forces and the internal energy. The formulations offered by the analytical mechanics have the advantage of easy configuration of the equations of motion and the elaboration of algorithms that make it possible to receive the motion equations describing the elastic evolution of the MBS. The paper presents, according to our knowledge for the first time for such systems, the Gibbs-Appell equations and the Hamilton equations, in comparison with the classical representation of Lagrange's equations.

2. FEA of elastic MBS

The method used especially in the study of MBS having elastic elements is represented by the Lagrange's equations. It is necessary, in a first step, to obtain the evolution equations for one elastic element. The shape functions determine the matrix coefficients of the obtained differential system that describe the dynamic response of an element. If we work in a local coordinate system this step is followed by reporting all the sizes used to the global reference frame, after which one can proceed to assembling the obtained equations systems in a general global system of equations that determines the dynamic response for the entire structure. These procedures are conducted according to the classical methods, applied in the FEA. By introducing the boundary conditions and the applied loads, one can proceed

to solving the obtained equation system. The deformations are considered small enough so the overall motion of the system considered as rigid body system is not influenced.

In the most general case, one, two- or three-dimensional finite elements can be used to study such a structure. The first studies on these systems have been applied to one-dimensional finite elements with plane motion [1–4]. The research was then extended to two-dimensional and three-dimensional systems [5, 6]. More sophisticated models have been developed over the last two decades [7–12].

To study the evolution equations of an element, when a one, two or three-dimensional elements is used, the Lagrange's equations has been applied in most studies. This method has proved to be a reliable, useful and verified calculation method. Its main advantage is that it uses the expressions of kinetic energy, mechanical work of the external forces and potential energy, notions with which the researchers are very familiar.

From a theoretical point of view, analytical mechanics offers a variety of methods to obtain the evolution equations of a mechanical system. Generally these methods introduce new notions, some of them rarely used by researchers. However, an analysis of these methods is required, as they may be useful for certain special situations. For this reason, in this study we will focus on three alternative approaches to the problem. The most common method used in engineering practice is the Lagrange method. The acceleration energy method proves to be another method that can present certain advantages for the MBS with elastic elements study. Finally is analyzed the Hamilton's equations method, which has a real advantage, the second order differential system of equations is reduced to a first order differential system but, at the same time, it has the disadvantage of intricate representations, with many terms to be considered.

We mention that the Lagrangian contains physical quantities that the researchers are familiar with (kinetic energy, potential, work) and, as a result, writing it is relatively easy. It is one of the main reasons why the Lagrangian equations method is so widely used. Another reason is the use of generalized coordinates, which allows the unitary treatment of the system and a formal representation that is well suited to numerical calculation. Eliminating the liaison forces is also strong point of the method, so the number of unknowns is greatly reduced. In the FEA where we work with systems having a high number of DOF, of the order of thousands or tens of thousands, this property leads to significant decrease regarding the computing effort and so, is useful for users.

The energy of accelerations presents some difficulties for the user, firstly because it is a notion with which the researchers are less familiar and secondly because the number of terms in the expression of acceleration

is five, while in the expression of velocity we have only four terms [13, 14]. It follows that the number of operations required to obtain the energy of accelerations is greater than the number of operations necessary to determine the kinetic energy. This disadvantage, however, is compensated by the fact that the number of differentiations required in the application of the Lagrange method will be much greater than in the case of the use of the acceleration energy. From an economical point of view, the computation time used will be reduced when applying the Gibbs-Appell equations. Although the method is less common, we can mention that in the last decade more researchers apply the method in their work [15–19]. This fact is due to the ease of obtaining the equations of motion for the studied mechanical systems. The great advantage of using Lagrange's equations is due to the fact that the Lagrangian contains the expression of kinetic energy and potential energy, two notions very familiar to the mechanical engineering researcher.

Regarding Hamilton's equations, we have no data showing the use of this method when applying FEA to MBS with elastic elements. In these circumstances, the present study represents a first attempt to use this method in FEA of such systems. Using Hamilton's equations may also be an option for analyzing MBS systems with FEM, but from our knowledge, such an approach has not been made so far and we will see that this is mainly due to the large number of calculations to be made.

3. Kinematics, kinetic energy, momentum, acceleration energy, Hamiltonian, potential energy and work of a finite element

3.1. Kinematics

In the following, a three-dimensional finite element considered as an elastic solid will be analyzed. In the approximation of the FEM, the displacement of a certain point of the considered solid is uniquely defined by a set of independent coordinates, through an interpolation function. These coordinates can be the nodal coordinates, their derivatives or other kinematic sizes.

The element will relate to a mobile coordinate frame. In the final analysis, all the finite elements will be related to a global coordinate system. The mobile reference system its movement is defined by the angular velocity and acceleration $\bar{\omega}(\omega_x, \omega_y, \omega_z)$ respectively $\bar{\varepsilon}(\varepsilon_x, \varepsilon_y, \varepsilon_z)$, velocity $\bar{v}_o(\dot{X}_o, \dot{Y}_o, \dot{Z}_o)$ and acceleration $\bar{a}_o(\ddot{X}_o, \ddot{Y}_o, \ddot{Z}_o)$ of the origin of the local reference system. All these vectors are considered known. Since we are working with two reference systems, one local and the other global, we will use two L and G indices to show this. The non-indexed sizes will be considered written in the local reference frame. In the rigid kinematics, the orthonormal operator [7] is used to change the components of

a vector from the mobile reference frame to the fixed one (global), $\{t\}_G = [T]\{t\}_L$. Differentiating the matrix $[T]$, it obtains the angular velocity and operator/vector and the angular acceleration operator/vector ([20]).

Let us now consider a point that belongs to the elastic finite element finite. Then this point, let's call it M , becomes, after deformation M' and it will have the position vector $\{r_{M'}\}_G$ defined by:

$$\{r_{M'}\}_G = \{r_O\}_G + [T](\{r\}_L + \{u\}_L) \quad (1)$$

$\{r_M\}_G$ represents the position vector of the point M , $\{u\}_L$ - the displacement vector and $\{r_O\}_G$ is the position vector of the origin of the local reference frame.

In FEA the continuous vector field of the displacements is approximated, according to the independent coordinates of the finite element, through a linear relation of the type:

$$\{u\}_L = [S]\{d\}_L \quad (2)$$

$\{d\}_L$ is the vector of independent coordinates. The velocity vector of M' it is:

$$\begin{aligned} \{v_{M'}\}_G &= \{\dot{r}_{M'}\}_G = \{\dot{r}_O\}_G + [\dot{T}]\{r\}_L + [T][\dot{S}]\{d\}_L \\ &+ [T][S]\{\dot{d}\}_L \end{aligned} \quad (3)$$

and acceleration vector:

$$\begin{aligned} \{a_{M'}\}_G &= \{\ddot{r}_O\}_G + [\ddot{T}]\{r\}_L + [T][\ddot{S}]\{d\}_L + 2[\dot{T}][S]\{\dot{d}\}_L \\ &+ [T][S]\{\ddot{d}\}_L. \end{aligned} \quad (4)$$

In the local coordinate system these vector becomes:

$$\begin{aligned} \{v_{M'}\}_L &= [T]^T \{v_{M'}\}_G = \{\dot{r}_O\}_L + [T]^T [\dot{T}]\{r\}_L \\ &+ [T]^T [T][S]\{\dot{d}\}_L + [S]\{\dot{d}\}_L \end{aligned} \quad (5)$$

$$\begin{aligned} \{a_{M'}\}_L &= [T]^T \{a_{M'}\}_G = \{\ddot{r}_O\}_L + [T]^T [\ddot{T}]\{r\}_L \\ &+ [T]^T [\ddot{T}][S]\{d\}_L + 2[T][\dot{T}][S]\{\dot{d}\}_L + [S]\{\ddot{d}\}_L \end{aligned} \quad (6)$$

3.2. Kinetic energy

An important element in all the formulations is the kinetic energy, obviously for its role as a prime integral. The expression of kinetic energy in the studied case is:

$$E_c = \frac{1}{2} \int_V \rho \{v_{M'}\}_G^T \{v_{M'}\}_G dV. \quad (7)$$

The kinetic energy can also be written in the global coordinate frame. Thus it is known that the transition of the independent coordinate vector $\{d\}_L$ to the global system, when it becomes $\{d\}_G$, is done through an

orthogonal matrix $[T]$ (see [6]):

$$\{d\}_G = [P]\{d\}_L. \quad (8)$$

We have too:

$$\{d\}_L = [P]^T \{d\}_G. \quad (9)$$

It is easy to obtain the new form of the kinetic energy using (8) and (9). The two expressions of kinetic energy are obviously equal.

Some notation will be used in the following:

$$[m] = \int_V \rho [S]^T [N] S dV \quad (10)$$

It will also be noted:

$$\begin{aligned} [m^i_O] &= \int_V \rho [S]^T dV; \{q^i(\varepsilon)\}_L \\ &= \int_V \rho [S]^T [\varepsilon]_L \{r\}_L dV; \{q^i(\omega)\}_L \\ &= \int_V \rho [S]^T [\omega]_L [\omega]_L \{r\}_L dV \end{aligned} \quad (11)$$

$$[k(\varepsilon)] = \int_V \rho [S]^T [\varepsilon] [S] dV; \quad (12)$$

$$[k(\omega)] = \int_V \rho [S]^T [\omega]_L [\omega]_L [S] dV; \quad (13)$$

$$[c] = \int_V \rho [S]^T [\omega]_L [S] dV; \quad (14)$$

$$\{m_{ix}\} = \int_V \rho [S_{(i)}]^T x dV; \quad \{m_{iy}\} = \int_V \rho [S_{(i)}]^T y dV;$$

$$\{m_{iz}\} = \int_V \rho [S_{(i)}]^T z dV. \quad (15)$$

3.3. Potential energy

The internal work (potential energy) has the well-known form:

$$E_p = \frac{1}{2} \int_V \{\sigma\}^T \{\varepsilon\} dV. \quad (16)$$

$\{\varepsilon\}$ contains the distinct component of the strains tensor; $\{\sigma\}$ contains the distinct component of the stress tensor.

The relation between stress and strain is offered by the Hooke law:

$$\{\sigma\} = [H]\{\varepsilon\}. \quad (17)$$

The differential relationships linking strains and finite deformations are [6]:

$$\{\varepsilon\} = [b]\{u\} = [b][S]\{d\} \quad (18)$$

Using (17) and (18) the internal work (16) becomes:

$$E_p = \frac{1}{2} \{d\}_L^T \left(\int_V [S]^T [b]^T [H]^T [b] [S] dV \right) \{d\}_L \quad (19)$$

With $[k]$ is denoted:

$$[k] = \int_V [S]^T [b]^T [H]^T [b] [S] dV \quad (20)$$

The potential energy takes the traditional form:

$$E_p = \frac{1}{2} \int_V \{d\}_L^T [k] \{d\}_L dV \quad (21)$$

3.4. Work

There are two types of loads that can cause an external mechanical work: concentrated forces acting in knots, $\{q\}_L$ and volume forces $\{p\} = \{p(x, y, z)\}$. The concentrated forces $\{q\}_L$ give a mechanical work:

$$W^c = \{q\}_L^T \{d\}_L \quad (22)$$

and the vector of volume forces, the work:

$$W = \int_V \{p\}_L^T \{f\}_L dV = \left(\int_V \{p\}_L^T [S] dV \right) \{d\}_L = \{q^*\}_L^T \{d\}_L \quad (23)$$

3.5. Lagrangian

The classical form of the Lagrangian is [21]:

$$L = E_c - E_p + W + W^c \quad (24)$$

Using (7),(21),(22),(23) it obtains:

$$\begin{aligned} L = & \frac{1}{2} \int_V \rho (\{\dot{r}_O\}_L^T \{\dot{r}_O\}_L + 2\{\dot{r}_O\}_L^T [T]^T [\dot{r}]_L) \\ & + 2\{\dot{r}_O\}_L^T [T]^T [\dot{r}]_L [S] \{d\}_L + 2\{\dot{r}_O\}_L^T [S] \{\dot{d}\}_L) dV \\ & + \frac{1}{2} \int_V \rho (\{r\}_L^T [\dot{r}]_L^T [\dot{r}]_L + 2\{r\}_L^T [\dot{r}]_L^T [\dot{r}]_L [S] \{d\}_L \\ & + 2\{r\}_L^T [\dot{r}]_L^T [T] [S] \{\dot{d}\}_L) dV \\ & + \frac{1}{2} \int_V \rho (\{d\}_L^T [S]^T [\dot{r}]_L^T [\dot{r}]_L [S] \{d\}_L \\ & + 2\{d\}_L^T [M]^T [\dot{r}]_L^T [T] [S] \{\dot{d}\}_L \\ & + \{\dot{d}\}_L^T [S]^T [S] \{\dot{d}\}_L) dV \\ & - \frac{1}{2} \int_V \{d\}_L^T [k] \{d\}_L dV + \{q^*\}_L^T \{d\}_L + \{q\}_L^T \{d\}_L \end{aligned} \quad (25)$$

3.6. Momentum

A way to calculate the momentum, if the Lagrangian is known, is to use the formula:

$$\{p\}_L = \left\{ \frac{\partial L}{\partial \{\dot{d}\}_L} \right\}. \quad (26)$$

From the expression of momentum (26) we can obtain the vector of velocities of the nodal displacement $\{\dot{d}\}_L$

3.7. Hamiltonian

The expression of Hamiltonian becomes, in our case:

$$H = \left\{ \frac{\partial L}{\partial \{\dot{d}\}_L} \right\}^T \{\dot{d}\}_L - L, \quad (27)$$

where for the Lagrangian is used the rel.(25).

3.8. Energy of accelerations

Definition: The energy of accelerations is defined, in the case of nV material points, as [22]:

$$E_a = \frac{1}{2} \sum_{i=1}^N m_i a_i^2. \quad (28)$$

If we consider a solid body the definition becomes:

$$E_a = \frac{1}{2} \int_V \rho a^2 dV \quad (29)$$

Using the rel.(4) to obtain acceleration, rel. (32) give us:

$$\begin{aligned} E_a = & \frac{1}{2} \int_V \rho a_{M'}^2 dV = \frac{1}{2} \int_V \rho \{a_{M'}\}^T \{a_{M'}\} dV \\ = & \frac{1}{2} \int_V \rho (\{\ddot{r}_O\}_G^T + \{r\}_L^T [\ddot{r}]_L^T + \{d\}_L^T [S]^T [\ddot{r}]_L^T \\ & + 2\{\dot{d}\}_L^T [S]^T [\dot{r}]_L^T + \{\ddot{d}\}_L^T [S]^T [T]^T) x \times (\{\ddot{r}_O\}_G \\ & + [\ddot{r}]_L \{r\}_L + [\ddot{r}]_L [S] \{d\}_L + 2[\dot{r}]_L [S] \{\dot{d}\}_L \\ & + [T] [S] \{\ddot{d}\}_L) dV \end{aligned} \quad (30)$$

In [22] a complete expression of this energy is made. This expression seems to be more complicated than the expression of kinetic energy but it has advantages which we will show in the paper.

4. Lagrange's equations

Lagrange's equations applied for a finite element are:

$$\frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{d}} \right\}_L - \left\{ \frac{\partial L}{\partial d} \right\}_L = 0 \quad (31)$$

By $\{\partial E / \partial X\}$ we have denoted:

$$\left\{ \frac{\partial E}{\partial X} \right\} = \left\{ \begin{array}{c} \frac{\partial E}{\partial x_1} \\ \frac{\partial E}{\partial x_2} \\ \vdots \\ \frac{\partial E}{\partial x_n} \end{array} \right\} \text{ where : } \{X\} = \left\{ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} \quad (32)$$

It obtains:

$$\begin{aligned} [m] \{\ddot{d}\}_L + [c] \{\dot{d}\}_L + ([k] + [k(\varepsilon)] + [k(\omega)]) \{d\}_L = & \{q\}_L \\ + \{q^*\}_L - \{q^i(\varepsilon)\}_L - \{q^i(\omega)\}_L - [m^i_O] \{\ddot{r}_O\}_L \end{aligned} \quad (33)$$

We note that we have to make three differentiations $\{\partial L / \partial d\}$, $(d/dt) \{\partial L / \partial \dot{d}\}$, $\{\partial L / \partial d\}$ whereas in the case of

applying the Gibbs-Appell equations it is only necessary to make a single differentiation $\{\partial E_a / \partial \ddot{d}\}$ [20]. This can reduce the computation time and thus an advantage when applying FEA to MBS with elastic elements.

5. Gibbs-Appell formalism

An alternative form of writing Lagrange's equations is the Gibbs-Appell equation. To use these it is necessary to write, first, the energy of acceleration, represented in our paper by the rel.(30). These equations are given by relations [22]:

$$\frac{\partial E_a}{\partial \ddot{q}_j} = Q_j \quad j = \overline{1, n}. \tag{34}$$

The equations (31) are composed by the following terms [23]:

- quadratic terms E_{a2} containing the vector of accelerations $\{\ddot{d}\}_L$, noted with E_{a2} :

$$E_{a2} = \frac{1}{2} \int_V \rho (\{\ddot{d}\}_L^T [S]^T [S] \{\ddot{d}\}_L) dV \tag{35}$$

- linear terms in accelerations E_{a1} :

$$E_{a1} = \int_V \rho (\{\ddot{d}\}_L^T [S]^T [T]^T \{\ddot{r}_O\} + \{\ddot{d}\}_L^T [S]^T [T]^T \{\ddot{r}\}_L + \{\dot{d}\}_L^T [S]^T [T]^T \{\dot{r}\}_L + 2\{\dot{d}\}_L^T [S]^T [T]^T \{\dot{r}\}_L) dV \tag{36}$$

- terms without accelerations E_{a0} . They will not play any role in obtaining the equations. These are represented by the rest of the term of the expression (31). Not showing an importance for the calculations that follow, we will not write them explicit.

Equation Gibbs-Appell (35) can be written:

$$\left\{ \frac{\partial E_a}{\partial \ddot{d}} \right\}_L - \{Q\}_L = 0 \tag{37}$$

where:

$$E_a = E_{a0}(\dot{q}) + E_{a1}(\dot{q}, \ddot{q}) + E_{a2}(\ddot{q}) \tag{38}$$

and

$$\{Q\}_L = [k]\{d\}_L + \{q\}_L + \{q^*\}_L. \tag{39}$$

We have:

$$\frac{\partial E_{a2}}{\partial \{\ddot{d}\}_L} = \left(\int_V \rho [N]^T [S] dV \right) \{\ddot{d}\}_L = [m]\{\ddot{d}\}_L; \tag{40}$$

$$\frac{\partial E_{a1}}{\partial \{\dot{d}\}_L} = -[m'_O]\{\ddot{r}_O\} - \{q^j(\omega)\} - \{q^j(\varepsilon)\} + ([k(\omega)] + [k(\varepsilon)])\{d\}_L + [c]\{\dot{d}\}_L; \tag{41}$$

$$\frac{\partial E_{a0}}{\partial \{\dot{d}\}_L} = 0. \tag{42}$$

Finally, is obtained the motion equations in the form (33).

6. Hamilton's method

Using Lagrange's method leads to a second order equations system. The unknowns are the generalized coordinates q_i . Technically, to solve the system of the second order with n unknowns, it is necessary to write it as a system the first order, with $2n$ unknowns, by introducing new unknowns which are the generalized velocities. Hamiltonian mechanics also uses $2n$ unknowns and the system of differential equations to be solved is a system of $2n$ equations of the first order. The first n unknowns in this case are the generalized coordinates, the other n being canonically conjugated moment, defined by:

$$\{p\}_L = - \left\{ \frac{\partial L}{\partial \{\dot{d}\}_L} \right\}. \tag{43}$$

The use of canonically conjugate momenta instead of generalized velocities represents an important difference between Lagrange's method and Hamilton's method. Hamilton's equations are given by relations [22]:

$$\{\dot{d}\}_L = \left\{ \frac{\partial H}{\partial \{p\}_L} \right\}; \quad \{\dot{p}\}_L = - \left\{ \frac{\partial H}{\partial \{d\}_L} \right\} \tag{44}$$

Considering the relations (26) - (27) it obtains:

$$\{\dot{d}\}_L = [m'_O]^{-1} (\{p\}_L - m\{\dot{r}_O\}_L - m[T]^T [\dot{T}]\{\dot{r}_C\}_L - [T]^T [\dot{T}][m'_O]\{d\}_L)$$

$$\begin{aligned} \{\dot{p}\}_L = & \{p\}_L^T [m'_O]^{-1} [T]^T [\dot{T}][m'_O] + \int_V \rho (\{\dot{r}_O\}_L^T [T]^T [\dot{T}][S]) dV \\ & + \int_V \rho (\{\dot{r}\}_L^T [\dot{T}][S]) dV + \int_V \rho ([S]^T [\dot{T}][S]) \{d\}_L \\ & + [S]^T [\dot{T}][T][S] \{\dot{d}\}_L dV - \int_V [k]\{d\}_L dV \\ & + \{q^*\}_L^T + \{q\}_L^T. \end{aligned} \tag{45}$$

7. Conclusions and discussions

The main step in the process of solving MBS systems with elastic elements is to write the motion equation for a chosen finite element. If these equations are written, then the other steps that follow, namely the assembly of the equations of motion and solving them will be done according to the classical methods currently used in the common FEM software. Writing these equations proves to be a difficult thing, given the many terms that appear in these equations. Therefore, finding a formalism that allows the easy writing of these equations is an important goal in this analysis. In the use of FEM in MBS systems with elastic elements, until now, the almost exclusive Lagrangian equation method has been used. This method presents the advantages of a homogeneous writing and the possibility to follow the calculus is easier. In the Introduction section, some

of these approaches were presented. From the theoretical point of view, however, there are several equivalent formulations, developed by analytical mechanics, equivalent to Lagrange's equations. You can use the Gibbs-Appell equations, Hamilton's equations, Maggi's equations, Jacobs's equations and other equivalent forms [24–26]. The question arises to what extent these formulations may be more advantageous than using Lagrange's equations. In the paper it is presented comparative, for the first time, three analytical formulations of this problem to see the advantages and disadvantages of such approaches, which could ultimately lead to a more economical writing of the equations of motion. Lagrange's equations show the major advantage of a frequent previous use of the method and of a regular use of the researchers with this method. Using the Gibbs-Appell method is proving to be more economical in terms of the time required to obtain the equations and the final results. The last method used, that of Hamilton's equations, is proving to be the most lucrative, the time required to obtain the equations not being economical and the complexity of the intermediate calculations being high. We do not deny, however, that this method could be useful in a new approach to solving the differential system since the system of differential equations obtained is of first order and thus a calculation step, used in the classical solving of these systems is avoided. If the Gibbs – Appell equations are used, the number of differentiations of the terms that need to be made decreases and the total number of calculations required also decreases compared to Lagrange's method. Lagrange's method, however, presents the advantage of using kinetic energy, a notion that we are used to and that we operate with ease. In the Gibbs-Appell case, the energy of accelerations is used, a notion most engineers are less familiar with.

In the future, it is presumed that the equivalent forms developed by the analytical mechanics and which apparently have no practical applicability will be re-evaluated and new fields of applicability will be found in the engineering studies in which they will prove their usefulness [27–37].

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