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# The Backward in Time Problem of Double Porosity Material with Microtemperature

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**Abstract:** In the present study, the theory of thermoelastodynamics is considered in the case of materials with double porosity structure and microtemperature. The novelty of this study consists in the investigation of a backward in time problem associated with double porous thermoelastic materials with microtemperature. In the first part of the paper, in case of the bounded domains the impossibility of time localization of solutions is obtained. This study is equivalent to the uniqueness of solutions for the backward in time problem. In the second part of the paper, a Phragmen-Lindelof alternative in the case of semi-infinite cylinders is obtained.

**Keywords:** thermoelasticity with double porosity; microtemperature; backward problem; impossibility of localization

## 1. Introduction

In recent years, many authors have been interested in the study of the thermoelastic bodies with double porosity structure. The first studies regarding the linear theory are encountered in the papers of Barenblatt et al. [1]. The theory of thermoelastic solids with double porosity structure consists in the study of two types of porosity: the macro porosity represented by the body's pores and the micro porosity represented by the fissures that appear in the skeleton. According to Barneblatt et al. [2], Berryman et al. [3] and Khalili et al. [4], some interesting applications of materials with double porosity structure are encountered in geophysics and according to Cowin, [5] in mechanics of bone. In the case of elastic materials with double porosity, the basic equations contain the displacement vector field, and two types of pressure associated with the macro and micro porosity [6–8]. The fluid pressure is independent of the displacement vector field in the case of equilibrium theory.

Nunziato and Cowin studied, in reference [9], the behaviour of the porous bodies when the pores are elastic and the fissures are void of material. The applications of this theory are encountered in geology in the study of rocks and soils, in the manufacture of ceramics and pressed powders. The theory of Nunziato and Cowin for the solids with voids was used by Iesan and Quintanilla [10] in order to develop a theory for thermoelastic materials with double porosity. The displacement vector field influences the porosity of material in the equilibrium case. For the linear thermoelastic solids with voids, Quintanilla in [11] derived the impossibility of the localization in time of the solutions.

The study of the backward in time problem is very important from the thermomechanical point of view, because it offers information about the behavior of the system in the past using the information that is available at the present time. Usually the Saint Venant's principle is used for the spatial behavior of the solutions for partial differential equations. The studies regarding the spatial decay estimates were obtained for elliptic [12], parabolic [13,14] and hyperbolic [15] equations. The main aim of the spatial decay estimates is to model the perturbations on a side of the boundary that are damped for the points located at some distance from this side of the boundary. For this analysis it is necessary to use a semi-infinite cylinder whose finite end is perturbed and the goal is to identify the effects when the

spatial variable increases. The harmonic vibrations in thermoelastic dynamics with double porosity structure for the backward in time problem was studied by Florea [16].

In the paper by Marin et al. [17], the unique problem of modeling a thermoelastic composite material with dipolar structure was studied. Bazzara et al. [18] analyzed some qualitative properties in case of solids with double porosity when the viscous dissipation is taken into consideration. Ieşan and Quintanilla, in reference [19], proved the uniqueness for the mixed boundary value problem in the case of double porosity structure material.

The study of the localization in time of solutions is represented by the phenomenon for which the mechanisms of dissipation are very strong, such that the solutions vanish after a finite time. The impossibility of localization in time of solutions is an open problem because the proof of this concept exists only in some linear situations. In the particular case of the linear thermodynamics theory of visco-elastic solids with voids, the solution's decay can be controlled by some particular exponential or polynomial functions [20–24]. The problem of the impossibility of localization of solutions was proved for the classical thermoelasticity with porous dissipation [25] and in the isotherm case with porous and elastic viscosity [11].

The aim of this paper is to show that in the case of thermoelasticity with double porosity structure and microtemperature, the only solution that vanishes after a finite time is the null solution, when the mechanisms of dissipation are the double porous dissipation, the temperature and the microtemperature. The obtained results can be also compared with those obtained in [20–24]. In the present paper, information regarding the upper bound for the solution decay will be presented. In the previous results [20–23], the authors proved that after a small period of time the thermomechanical deformations are very small and they can be neglected. In this paper it will be highlighted that they are not null for any positive time. The present study represents a continuation of the research regarding the impossibility of localization in thermo-porous-elasticity with microtemperatures realized by Quintanilla [26], using the results of Florea, [27]. To the best of my knowledge, this is the first work dealing with the study of the backward in time problem in case of double porosity solids with microtemperatures for anisotropic and homogeneous material.

This paper has two goals:

- in the first part of the paper, in case of the bounded domains the impossibility of time localization of solutions is obtained; this study is equivalent to the uniqueness of solutions for the backward in time problem for the case of double porous materials with microtemperature.
- in the second part of the paper, a Phragmen-Lindelof alternative in the case of semi-infinite cylinders is obtained.

The present study is structured as follows: in the Section 2 the basic equations for the backward in time problem in the case of materials with double porosity structure and microtemperature are described. Also, in this section the conditions imposed on the parameters that influence the behavior of the porous materials are presented. The impossibility of localization in time of solutions for the backward in time problem for a double porous material with microtemperature is expressed in the third section. It is stated here the conservation of the energy law and it is highlighted the main theorem of the present study. For the particular case of a semi-infinite cylinder a Phragmen-Lindelof alternative is obtained in Section 4. In the last section of the paper are drawn the conclusions of the present study.

## 2. Basic Equations for the Double Porous Materials with Microtemperature

In this study an open domain  $D$  of the three-dimensional space occupied by a body with a double porosity structure is considered. By  $\bar{D}$  is denoted the closure of  $D$  and the smooth piecewise boundary of the considered domain is noted by  $\partial D$ .

The behavior of a thermoelastic body with double porosity structure is characterized by the following functions that depends on the material point  $x \in D$  and the time variable  $t \in [0, t_0]$ :  $u_i$  the displacement vector field over  $D$ ,  $\phi$ ,  $\psi$  the volume fraction fields in the reference configuration.

The equations of evolution that govern the problem of thermoelasticity with double porosity structure for the materials with microtemperature in the absence of the supply terms are, [20,21]:

$$\begin{aligned}t_{ji,j} &= \rho \ddot{u}_i \\ \sigma_{j,j} + \xi &= k_1 \ddot{\phi} \\ \tau_{j,j} + \zeta &= k_2 \ddot{\psi}\end{aligned}\quad (1)$$

where:  $t_{ji}$  are the stress tensors,  $\rho$  is the mass density,  $\xi, \zeta$  are the intrinsic equilibrated body forces,  $k_1, k_2$  are the coefficients of equilibrated inertia.

The equation of energy is:

$$\rho T_0 \dot{\eta} = Q_{j,j} \quad (2)$$

The equation of the first moment of energy is given by the following relation:

$$\rho \dot{\varepsilon}_i = Q_{ji,j} + Q_i - q_i \quad (3)$$

In the equations below there are the following:  $\eta$  is the entropy,  $T_0$  is the constant absolute temperature of the body in the reference configuration,  $Q_j$  is the heat flux,  $\varepsilon_i$  represent the first moment of energy vector,  $Q_{ji}$  is the first heat flux moment tensor and  $q_i$  is the microheat flux average.

It is considered in this study that the material is centrosymmetric. In the context of the theory of the homogeneous thermoelastic bodies with double porosity structure and microtemperature the constitutive equations are:

$$\begin{aligned}t_{ij} &= C_{ijkl}u_{k,l} + B_{ij}\phi + D_{ij}\psi - \beta_{ij}\theta \\ \sigma_i &= \alpha_{ij}\phi_{,j} + b_{ij}\psi_{,j} - N_{ij}T_j \\ \tau_i &= b_{ji}\phi_{,j} + \gamma_{ij}\psi_{,j} - M_{ij}T_j \\ \xi &= -B_{ij}u_{i,j} - \alpha_1\phi - \alpha_3\psi + \gamma_1\theta \\ \zeta &= -D_{ij}u_{i,j} - \alpha_3\phi - \alpha_2\psi + \gamma_2\theta \\ \rho\eta &= \beta_{ij}u_{i,j} + \gamma_1\phi + \gamma_2\psi + a\theta \\ Q_i &= \kappa_{ij}\theta_{,j} + L_{ij}T_j \\ \rho\varepsilon_i &= -N_{ij}\phi_{,j} - M_{ji}\psi_{,j} - P_{ij}T_j \\ Q_{ij} &= -A_{ijrs}T_{s,r} \\ q_i &= (L_{ij} - R_{ij})T_j + (\kappa_{ij} - \lambda_{ij})\theta_{,j}\end{aligned}\quad (4)$$

where  $C_{ijkl}$  is the elasticity tensor,  $\beta_{ij}$  is the thermal dilatation tensor,  $\kappa_{ij}$  is the heat conductivity tensor,  $\beta_{ij}$  is the tensor of thermal dilatation,  $B_{ij}, D_{ij}, \alpha_{ij}, b_{ij}, \gamma_{ij}, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, a$  are typical functions in double porous theory and  $N_{ij}, M_{ij}, R_{ij}, \lambda_{ij}, A_{ijrs}$  are tensors which are usual in the theories with microtemperatures. In the constitutive Equation (4)  $\theta$  represents the temperature and  $T_i$  are the microtemperatures.

Introducing (4) in (1) the system of the field equations for the thermoelasticity with double porosity and microtemperatures is obtained:

$$\rho \ddot{u}_i = \left( C_{jikl}u_{k,l} + B_{ji}\phi + D_{ij}\psi - \beta_{ij}\theta \right)_{,j} \quad (5a)$$

$$k_1 \ddot{\phi} = \left( \alpha_{ij}\phi_{,i} + b_{ij}\psi_{,i} - N_{ij}T_j \right)_{,j} - B_{ij}u_{i,j} - \alpha_1\phi - \alpha_3\psi + \gamma_1\theta \quad (5b)$$

$$k_2 \ddot{\psi} = \left( b_{ij}\phi_{,i} + \gamma_{ij}\psi_{,i} - M_{ij}T_j \right)_{,j} - D_{ij}u_{i,j} - \alpha_3\phi - \alpha_2\psi + \gamma_2\theta \quad (5c)$$

$$a\dot{\theta} = -\beta_{ij}\dot{u}_{i,j} - \gamma_1\dot{\phi} - \gamma_2\dot{\psi} + \frac{1}{T_0} (\kappa_{ij}\theta_{,j} + L_{ij}T_j)_{,j} \tag{5d*}$$

$$P_{ij}\dot{T}_j = (A_{ijrs}T_{s,r})_{,j} - R_{ij}T_j - \lambda_{ij}\theta_{,j} - N_{ij}\dot{\phi}_{,j} - M_{ij}\dot{\psi}_{,j} \tag{5e*}$$

Proving the uniqueness of the solution of the backward in time problem, implies the impossibility of localization of the solutions of the above system. The system of equations which describes the backward in time problem is given by the same set of equations as (5a)–(5c) while (5d\*) and (5e\*) change into:

$$a\dot{\theta} = -\beta_{ij}\dot{u}_{i,j} - \gamma_1\dot{\phi} - \gamma_2\dot{\psi} - \frac{1}{T_0} (\kappa_{ij}\theta_{,j} + L_{ij}T_j)_{,j} \tag{5d}$$

$$P_{ij}\dot{T}_j = - (A_{ijrs}T_{s,r})_{,j} + R_{ij}T_j + \lambda_{ij}\theta_{,j} - N_{ij}\dot{\phi}_{,j} - M_{ij}\dot{\psi}_{,j} \tag{5e}$$

Because the constitutive coefficients are symmetric it results that:

$$C_{ijkl} = C_{klij}; \alpha_{ij} = \alpha_{ji}; b_{ij} = b_{ji}; B_{ij} = B_{ji}, D_{ij} = D_{ji}.$$

For the case of anisotropic and homogeneous material the tensors  $A_{ijrs}, P_{ij}, N_{ij}, M_{ij}, L_{ij}, R_{ij}, \lambda_{ij}$  are also symmetric:

$$A_{ijkl} = A_{lkij}, P_{ij} = P_{ji}, M_{ij} = M_{ji}, L_{ij} = L_{ji}, N_{ij} = N_{ji}, R_{ij} = R_{ji}, \lambda_{ij} = \lambda_{ji}.$$

In the context of theories with microtemperature as a consequence of Clausius-Duhem inequality, the following assumption is made, [20]:

$$\kappa_{ij}\theta_{,i} + (L_{ij} + T_0\lambda_{ij}) \theta_{,j}T_i + T_0R_{ij}T_iT_j + T_0A_{ijrs}T_{i,j}T_{s,r} \geq 0 \tag{6}$$

In order to obtain the estimated results it is necessary to impose the positivity of several functions and tensors:

$$\begin{aligned} \rho(X) \geq \rho_0 > 0; \quad k_1(X) \geq k_0^1 > 0; \quad k_2(x) \geq k_0^2 > 0; \\ a(x) \geq a_0 > 0; \quad P_{ij}\xi_i\xi_j \geq p_0\xi_i\xi_i, p_0 > 0 \end{aligned} \tag{a1}$$

$$\kappa_{ij}\xi_i\xi_j + (L_{ij} + T_0\lambda_{ij})\xi_j\xi_i + T_0R_{ij}\xi_i\xi_j \geq C_0(\xi_i\xi_i + \xi_j\xi_j), C_0 > 0, (\forall)\xi_i\xi_j \tag{a2}$$

$$\begin{aligned} C_{ijkl}u_{i,j}u_{k,l} + \alpha_{ij}\phi_{,i}\phi_{,j} + \gamma_{ij}\psi_{,i}\psi_{,j} + 2b_{ij}\phi_{,i}\psi_{,j} + 2B_{ij}u_{i,j}\phi + 2D_{ij}u_{i,j}\psi + \\ + \alpha_1\phi^2 + \alpha_2\psi^2 + 2\alpha_3\phi\psi \geq C^* \left( u_{i,j}u_{i,j} + \phi_{,i}\phi_{,i} + \psi_{,i} + \psi_{,i} + \phi^2 + \psi^2 \right), \end{aligned} \tag{a3}$$

$$\begin{aligned} \alpha_{ij}\xi_i\xi_j \geq 0; \quad b_{ij}\xi_i\xi_j \geq 0, (\forall)\xi_i \\ A_{jirs}\xi_{ij}\xi_{sr} \geq C_1\xi_{ij}\xi_{ij}, C_1 > 0, (\forall)\xi_{ij} \end{aligned} \tag{a4}$$

The assumption (a1) is related to the thermomechanical characteristics, (a2) and (a4) are consequences of the Clausius-Duhem inequality, (a3) gives the information that the internal energy is positive and may be expressed based on the theory of mechanical stability.

### 3. Main Results Regarding the Impossibility of Localization in Time

Let us consider a bounded domain  $B$  with the boundary  $\partial B$ . The study of impossibility of localization in time for solutions of the backward in time problem is equivalent with the study of the uniqueness of solutions for the mentioned problem given by the system of Equations (5a)–(5e). For the backward in time problem for the solids with double porosity structure and microtemperatures in order to prove the uniqueness of solutions it is sufficient to show that only the null solution satisfies

the problem with null initial and boundary conditions. In the next computations it is assumed that the domain  $B$  is smooth enough to apply the divergence theorem.

The initial conditions are:

$$\begin{aligned}
 u_i(\mathbf{X}, 0) = \dot{u}_i(\mathbf{X}, 0) = \phi(\mathbf{X}, 0) = \dot{\phi}(\mathbf{X}, 0) = 0 \\
 \psi(\mathbf{X}, 0) = \dot{\psi}(\mathbf{X}, 0) = \theta(\mathbf{X}, 0) = 0, \quad T_i(\mathbf{X}, 0) = 0 \quad \mathbf{X} \in B
 \end{aligned}
 \tag{7}$$

and the boundary conditions:

$$u_i(\mathbf{X}, t) = \phi(\mathbf{X}, t) = \psi(\mathbf{X}, t) = \theta(\mathbf{X}, t) = T_i(\mathbf{X}, 0) = 0, \quad \mathbf{X} \in \partial B, t \geq 0
 \tag{8}$$

The aim of this section is to obtain the energy relation for the double porous material with microtemperature. Multiplying (5a) by  $\dot{u}_i$ , (5b) by  $\dot{\phi}$ , (5c) by  $\dot{\psi}$ , (5d) by  $\theta$  and (5e) by  $T_j$ , the obtained relations will be integrated on  $[0, t]$  and they will be summed. Using the principle of conservation of energy, the divergence theorem and the boundary conditions, the following relation is obtained:

$$\begin{aligned}
 E_1(t) = \frac{1}{2} \int_B \left( \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a \theta^2 + P_{ij} T_j T_j + C_{ijkl} u_{i,j} u_{k,l} + 2B_{ij} \phi u_{i,j} + \right. \\
 \left. + 2D_{ij} \psi u_{i,j} + \alpha_{ij} \phi_{,j} \phi_{,j} + \gamma_{ij} \psi_{,j} \psi_{,j} + 2b_{ij} \psi_{,j} \phi_{,j} + \alpha_1 \phi^2 + 2\alpha_3 \phi \psi + \alpha_2 \psi^2 \right) dV = \\
 = \int_0^t \int_B \left[ \frac{1}{T_0} (\kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} T_i \theta_{,j}) + A_{ijrs} T_{s,r} T_{i,j} + R_{ij} T_i T_j + \lambda_{ij} \theta_{,j} T_i \right] dV ds
 \end{aligned}
 \tag{9}$$

Using the same procedure of multiplying the Equations (5a) by  $\dot{u}_i$ , (5b) by  $\dot{\phi}$ , (5c) by  $\dot{\psi}$ , (5d) by  $-\theta$  and (5e) by  $-T_j$ , integrating on  $[0, t]$  and using the divergence theorem the following expression is derived:

$$\begin{aligned}
 E_2(t) = \frac{1}{2} \int_B \left( \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 - a \theta^2 - P_{ij} T_j T_j + C_{ijkl} u_{k,l} u_{i,j} + 2B_{ij} \phi u_{i,j} + 2D_{ij} \psi u_{i,j} + \right. \\
 \left. + \alpha_{ij} \phi_{,j} \phi_{,j} + 2b_{ij} \psi_{,j} \phi_{,j} + \alpha_1 \phi^2 + 2\alpha_3 \phi \psi + \gamma_{ij} \psi_{,j} \psi_{,j} + \alpha_2 \psi^2 \right) dV = \\
 = - \int_0^t \left[ \frac{1}{T_0} \int_B (\kappa_{ij} \theta_{,j} \theta_{,j} + L_{ij} T_j \theta_{,j} + A_{ijrs} T_{s,r} T_{j,i} + R_{ij} T_j T_j + \lambda_{ij} \theta_{,j} T_j) dV \right] ds + \\
 + \int_0^t \int_B \left[ (\beta_{ij} \theta)_{,j} \dot{u}_i - (M_{ij} T_j) \dot{\psi} - (N_{ij} T_j) \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds
 \end{aligned}
 \tag{10}$$

Taking into consideration the Equations (5a)–(5e), the initial and boundary conditions, (7), (8), the following identity is obtained:

$$\begin{aligned}
 \int_B \left( \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 - a \theta^2 - P_{ij} T_j T_j \right) dV = \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + \alpha_{ij} \phi_{,i} \phi_{,j} + \gamma_{ij} \psi_{,i} \psi_{,j} + 2b_{ij} \psi_{,j} \phi_{,i} + \right. \\
 \left. + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \right) dV
 \end{aligned}
 \tag{11}$$

The impossibility of localization of the solutions in the theory with double porosity and microtemperature is proved in the following theorem.

**Theorem 1.** Let  $(u_i, \phi, \psi, \theta, T_i)$  be a solution of the backward in time problem (5a)–(5e) with the initial conditions (7) and the boundary conditions (8). The only solution of the mentioned problem is the null solution  $u_i = 0, \phi = 0, \psi = 0, \theta = 0, T_i = 0$ .

**Proof.** Replacing (11) into (10) it is obtained a new expression for  $E_2(t)$ :

$$\begin{aligned} E_2(t) &= \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + \alpha_{ij} \phi_{,i} \phi_{,j} + \gamma_{ij} \psi_{,i} \psi_{,j} + 2b_{ij} \phi_{,i} \psi_{,j} + 2B_{ij} u_{i,j} \phi + \right. \\ &\quad \left. + 2D_{ij} u_{i,j} \psi + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \right) dV = \\ &= - \int_0^t \int_B \frac{1}{T_0} \left( \kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} \theta_{,j} T_i + T_0 A_{ijrs} T_{s,r} T_{j,i} + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} \theta_{,j} T_i \right) dV ds + \\ &\quad + \int_0^t \int_B \left[ (\beta_{ij} \theta)_{,j} - (M_{ij} T_i)_{,j} \dot{\psi} - (N_{ij} T_i)_{,j} \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds \end{aligned}$$

The energy can be expressed under the bellow form, for a positive constant  $\varepsilon$ , small enough:

$$E(t) = E_2(t) + \varepsilon E_1(t), \quad \varepsilon \in (0, 1)$$

Taking into account that  $E(t)$  is a positive function, the energy can be expressed as follows:

$$\begin{aligned} E(t) &= \frac{\varepsilon}{2} \int_B \left( \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a \theta^2 + P_{ij} T_i T_j \right) dV + \\ &\quad + \frac{2+\varepsilon}{2} \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + \alpha_{i,j} \phi_{,i} \phi_{,j} + \gamma_{ij} \psi_{,i} \psi_{,j} + 2b_{ij} \phi_{,i} \psi_{,j} + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + \right. \\ &\quad \left. + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \right) dV \end{aligned}$$

On the other hand,

$$\begin{aligned} E(t) &= - \int_0^t \int_B \frac{1}{T_0} \left( \kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} \theta_{,j} T_i + T_0 A_{ijrs} T_{s,r} T_{j,i} + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} \theta_{,j} T_i \right) dV ds + \\ &\quad + \int_0^t \int_B \left[ (\beta_{ij} \theta)_{,j} \dot{u}_i - (M_{ij} T_i)_{,j} \dot{\psi} - (N_{ij} T_i)_{,j} \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds + \\ &\quad + \varepsilon \int_0^t \int_B \frac{1}{T_0} \left( \kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} T_i \theta_{,j} + T_0 A_{ijrs} T_{s,r} T_{j,i} + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_{,j} \right) dV ds \end{aligned}$$

The above relation yields, for  $\varepsilon \in (0, 1)$ :

$$\begin{aligned} E(t) &= - (1 - \varepsilon) \int_0^t \int_B \frac{1}{T_0} \left( \kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} \theta_{,j} T_i + T_0 A_{ijrs} T_{s,r} T_{j,i} + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_{,j} \right) dV ds + \\ &\quad + \int_0^t \int_B \left[ (\beta_{ij} \theta)_{,j} \dot{u}_i - (M_{ij} T_i)_{,j} \dot{\psi} - (N_{ij} T_i)_{,j} \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds \end{aligned}$$

from where:

$$\begin{aligned} \frac{dE(t)}{dt} &= - (1 - \varepsilon) \int_B \frac{1}{T_0} \left( \kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} \theta_{,j} T_i + T_0 A_{ijrs} T_{s,r} T_{j,i} + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_{,j} \right) dV ds + \\ &\quad + \int_B \left[ (\beta_{ij} \theta)_{,j} \dot{u}_i - (M_{ij} T_i)_{,j} \dot{\psi} - (N_{ij} T_i)_{,j} \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds \end{aligned}$$

but,

$$\int_B (\beta_{ij}\theta)_{,j} \dot{u}_i dV = \int_B \beta_{ij,j} \theta \dot{u}_i dV + \int_B \beta_{ij} \theta_{,j} \dot{u}_i dV$$

The inequality of arithmetic and geometric means implies that:

$$\int_B (\beta_{ij}\theta)_{,j} \dot{u}_i dV \leq C_1 \int_B (\rho \dot{u}_i \dot{u}_i + a\theta^2) dV + \varepsilon_1 \int_B \kappa_{ij} \theta_{,i} \theta_{,j} dV$$

where  $\varepsilon_1$  is small enough,  $C_1$  is a positive constant that can be determined based on the constitutive coefficients and  $\varepsilon_1$ ;

$$\int_B (M_{ij}T_i)_{,j} \dot{\psi} dV \leq C_2 \int_B (k_2 \dot{\phi}^2 + P_{ij}T_i T_j) dV$$

where  $C_2$  can be determined. Therefore, there is a positive constant  $C$  such that:

$$\frac{dE}{dt} \leq C \int_B (\rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a\theta^2 + P_{ij}T_i T_j) dV$$

which is equivalent with the estimate:

$$\frac{dE}{dt} \leq C^* E(t) \Leftrightarrow \frac{dE}{E} \leq C^* dt \Leftrightarrow \ln E \leq C^* t + C \Leftrightarrow E(t) \leq C e^{C^* t}.$$

The estimate:

$$E(t) \leq E(0)e^{C^* t}$$

is valid for  $t = 0$ . However, the initial condition leads us to  $E(t) = 0$  for every  $t \geq 0$  that is equivalent with:

$$\dot{u}_i = 0; \dot{\phi} = 0; \dot{\psi} = 0; \theta = 0; T_i(t) = 0 \Leftrightarrow u_i = C_1; \phi = C_2; \psi = C_3; \theta = T_i = 0$$

taking into account the initial conditions (7) it is obtained that the solution for the considered problem is the null solution:

$$u_i = 0; \phi = 0; \psi = 0; \theta = 0; T_i = 0$$

□

#### 4. Phragmen-Lindelof Alternative for the Solution of Backward in Time Problem with Double Porosity and Microtemperature

A semi-infinite prismatic cylinder  $B = D \times [0, \infty)$  is considered, which is occupied by a body with a double porosity structure with micro-temperature.  $D$  denotes the cross section in the cylinder. The boundary of the section is a piece-wise continuously differentiable curve denoted by  $\partial D$  sufficiently smooth to admit application of divergence theorem. The lateral surface of the cylinder is  $\Pi = \partial D \times (0, \infty)$ . The cylinder is assumed to be free of load on the lateral boundary surface.

The lateral boundary conditions are:

$$u_i(\mathbf{X}, t) = 0; \phi(\mathbf{X}, t) = 0; \psi(\mathbf{X}, t) = 0; \theta(\mathbf{X}, t) = 0; T_i(\mathbf{X}, T) = 0(\mathbf{X}, t) \in \Pi \times (0, \infty) \tag{12}$$

On the base of the cylinder the following boundary conditions are assumed:

$$\begin{aligned} u_i(x_1, x_2, 0, t) &= \tilde{u}_i; \phi(x_1, x_2, 0, t) = \tilde{\phi}; \\ \psi(x_1, x_2, 0, t) &= \tilde{\psi}; \theta(x_1, x_2, 0, t) = \tilde{\theta}; T_i(x_1, x_2, 0, t) = \tilde{T}_i \end{aligned} \tag{13}$$

For the solution of the problem determined by the system (5a)–(5e) with initial condition (12) and boundary condition (13) the goal is to obtain a Phragmen-Lindelof alternative necessary for the interpretation of the behavior of the solution of the boundary value problem. The main aim in this section is to estimate the absolute value of the defined function  $H_\omega$  from (14) by means of its spatial derivative.

Given the function:

$$\begin{aligned}
 H_\omega(z, t) = & \int_0^t \int_{D(z)} e^{-2\omega s} [C_{i3kl}u_{k,l} + B_{i3}\phi + D_{i3}\psi - \beta_{3i}\theta] \dot{u}_i dads + \\
 & + \int_0^t \int_{D(z)} e^{-2\omega s} [\alpha_{i3}\phi_{,i} + b_{i3}\psi_{,i} - N_{i3}T_i] \dot{\phi} dads + \\
 & + \int_0^t \int_{D(z)} e^{-2\omega s} [b_{3i}\phi_{,i} + \gamma_{i3}\psi_{,i} - M_{i3}T_i] \dot{\psi} dads + \\
 & + \int_0^t \int_{D(z)} e^{-2\omega s} \frac{1}{T_0} [\kappa_{i3}\theta_{,i} + L_{i3}T_i] \dot{\theta} dads \\
 & + \int_0^t \int_{D(z)} e^{-2\omega s} (A_{3irs}T_{s,r} + R_{i3}T_i + \lambda_{ij}\theta_{,i}) T_i dads
 \end{aligned} \tag{14}$$

Here  $D(z) = \{\mathbf{X} \in B | x_3 = z\}$  denotes the cross section of the cylinder at a distance  $z$  from the base. Through means of the divergence theorem and employing the field equations, boundary and initial conditions is obtained:

$$H_\omega(z + h, t) - H_\omega(z, t) = \frac{1}{2} \int_{R(z+h,z)} \chi_\omega(t) dV, (\forall) h > 0 \tag{15}$$

where  $R(z + h, z) = \{\mathbf{X} \in B | z < x_3 < z + h\}$ .

The internal energy is:

$$\begin{aligned}
 \Phi = & \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a\theta^2 + P_{ij}T_j T_j + C_{ijkl}u_{i,j}u_{k,l} + +2B_{ij}u_{i,j}\phi + 2D_{ij}u_{i,j}\psi + \\
 & + \alpha_{ij}\phi_{,i}\phi_{,j} + \gamma_{ij}\psi_{,i}\psi_{,j} + 2b_{ij}\phi_{,i}\phi_{,j} + \alpha_1\phi^2 + \alpha_2\psi^2 + 2\alpha_3\phi\psi
 \end{aligned} \tag{16}$$

such that:

$$\begin{aligned}
 \chi_\omega(t) = & e^{-2\omega t} \Phi(t) + \int_0^t e^{-2\omega s} \left[ 2\omega\Phi(s) + 2\frac{\kappa_{ij}}{T_0}\theta_{,i}(s)\theta_{,j}(s) + 2\frac{L_{ij}}{T_0}\theta_{,i}T_j(s) \right. \\
 & \left. + 2A_{ijrs}T_{s,r}(s)T_{i,j}(s) + 2R_{ij}T_i(s)T_j(s) + 2\lambda_{ij}\theta_{,i}T_j(s) \right] ds
 \end{aligned} \tag{17}$$

From (15) it results:

$$\frac{\partial H_\omega}{\partial z} = \frac{1}{2} \int_{D(z)} \chi_\omega(t) dz$$

that leads to the following relation:

$$\frac{\partial H_\omega}{\partial z} = \frac{e^{-2\omega t}}{2} \int_{D(z)} \Phi(t) da + \int_0^t \int_{D(z)} e^{-2\omega s} [\omega\Phi(s) + W] dads \tag{18}$$

where,

$$W = \frac{\kappa_{ij}}{T_0} \theta_{,i} \theta_{,j} + \frac{L_{ij}}{T_0} \theta_{,i} T_j + A_{ijrs} T_{s,r} T_{i,j} + R_{ij} T_i T_j + 2\lambda_{ij} \theta_{,i} T_j$$

Further, the aim is to estimate the absolute value of  $H_\omega$  in terms of spatial derivatives, in order to get a differential inequality, such that:

$$|H_\omega| \leq C_\omega \frac{\partial H_\omega}{\partial z}, \quad (\forall) z \geq 0 \tag{19}$$

The above inequality is known in the literature of specialty regarding the spatial estimate as Phragmén-Lindelöf alternative.

Under the assumption (a3) the internal energy from (16) leads us to the following inequality:

$$\Phi \geq \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a\theta^2 + P_{ij} T_i T_j + C^* (u_{i,j} u_{i,i} + \phi_{,i} \phi_{,i} + \psi_{,i} \psi_{,i} + \phi^2 + \psi^2)$$

Therefore the relation (15) yields:

$$\begin{aligned} & H_\omega(z+h, t) - H_\omega(z, t) \geq \\ & \geq \frac{1}{2} \int_R e^{-2\omega s} \left[ \rho \dot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a\theta^2 + P_{ij} T_i T_j + C^* (u_{i,j} u_{i,i} + \phi_{,i} \phi_{,i} + \psi_{,i} \psi_{,i} + \phi^2 + \psi^2) \right] da + \\ & + \int_0^t \int_R e^{-2\omega s} \left\{ \omega \left[ k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a\theta^2 + P_{ij} T_i T_j + C^* (u_{i,j} u_{i,i} + \phi_{,i} \phi_{,i} + \psi_{,i} \psi_{,i} + \phi^2 + \psi^2) \right] + \kappa_{ij} \theta_{,i} \theta_{,j} \right. \\ & \left. + \frac{\kappa_{ij}}{T_0} \theta_{,i} \theta_{,j} + \frac{L_{ij}}{T_0} \theta_{,i} T_j + A_{ijrs} T_{i,j} T_{r,s} + R_{ij} T_i T_j + \lambda_{ij} \theta_{,i} T_j \right\} dads \end{aligned}$$

Based on the inequality of arithmetic and geometric means and also the Cauchy-Schwarz inequality it results that:

$$\begin{aligned} |H_\omega(z, t)| \leq C_\omega & \left[ \frac{e^{-2\omega t}}{2} \int_{D(z)} \Phi(t) dz + \int_0^t \int_{D(z)} \omega e^{-2\omega s} \Phi(s) dads + \right. \\ & \left. + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ \frac{1}{T_0} (\kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} T_i \theta_{,j}) + A_{ijrs} T_{i,j} T_{r,s} + R_{ij} T_i T_j + \lambda_{ij} T_i \theta_{,j} \right] dads \right] \end{aligned}$$

Thus the alternative (19) was proved.

From the inequality (19) the following two inequalities can be extracted:

$$-\frac{\partial H_\omega}{\partial z} \leq \frac{1}{C_\omega} H_\omega \text{ and } \frac{\partial H_\omega}{\partial z} \geq \frac{1}{C_\omega} H_\omega \tag{20}$$

Taking into consideration the computations from Flavin et al., [15] two estimates are obtained:

$$H_\omega(z, t) \geq H_\omega(z_0, t) e^{\frac{z-z_0}{C_\omega}} \tag{21}$$

$(\forall) z \geq z_0, z_0 > 0$  and  $H_\omega(z_0, t) > 0$  that lead to:  $\lim_{z \rightarrow \infty} e^{-\frac{z}{C_\omega}} \int_{R(z)} \chi_\omega(t) dv > 0$  and

$$-H_\omega(z, t) \leq H_\omega(z_0, t) e^{\frac{-z}{C_\omega}} \tag{22}$$

$(\forall)z \geq 0$  and  $H_\omega(z, t) \leq 0$ . From (22) it is obvious that  $H_\omega(z, t) \rightarrow 0$  for  $z \rightarrow \infty$ .

Let us introduce the following estimate:

$$E_\omega(z, t) = \frac{e^{-2\omega t}}{2} \int_{R(z)} \Phi(t) dz + \int_0^t \int_{R(z)} e^{-2\omega s} \left[ \omega \Phi(s) + \frac{1}{T_0} (\kappa_{ij} \theta_{,i} \theta_{,j} + L_{ij} T_i \theta_{,j}) + A_{ijrs} T_{i,j} T_{r,s} + R_{ij} T_i T_j + \lambda_{ij} T_i \theta_{,j} \right] dads \quad (23)$$

where  $R(z) = \{X \in B | z < x_3\}$ . Based on (22) it is observed that:

$$E_\omega(z, t) \leq E_\omega(0, t) e^{-\frac{z}{c_\omega}}, z \geq 0 \quad (24)$$

Now, the following conclusions can be drawn: if  $(u_i, \phi, \psi, \theta, T_i)$  is a solution of the backward in time problem defined by the system (5a)–(5e) with the null initial conditions (7) and boundary conditions (8) there are two situations: the solution satisfies the asymptotic condition:  $\lim_{z \rightarrow \infty} e^{-\frac{z}{c_\omega}} \int_{R(z)} \chi_\omega(t) dv > 0$  or it satisfies the decay estimate (22). This study can continue with obtaining of the upper bound for the amplitude  $E_\omega(0, t)$  in terms of the boundary conditions, but this analysis will be the subject of another paper.

## 5. Conclusions

In this paper, two goals were achieved: the impossibility of time localization of solutions in case of the bounded domains and a Phragmen-Lindelof alternative in the case of semi-infinite cylinders were obtained. To the best of my knowledge this is the first study dealing with these two problems in the context of thermoelasticity of double porosity for anisotropic and homogeneous materials with microtemperatures.

The uniqueness of the solutions for the backward in time problem in case of the materials with double porosity structure with microtemperature was proved. The conclusion that for the backward in time problem the only solution that vanishes is the null solution for every  $t > 0$  is drawn. In the case of linear thermoelastic theories, these results cannot certify that the thermomechanical deformations from double porous bodies with microtemperature vanish after a finite time. In this situation, it is necessary that time should be unbounded to guarantee that the fraction of volumes becomes the same as the reference configuration. A function that defines a measure on the solutions was obtained and the usual exponential type alternative for the solutions of the problem defined in a semi-infinite cylinder was deduced.

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