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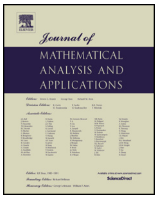

Regular Articles

A very general framework for fractal interpolation functions

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A R T I C L E I N F O

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A B S T R A C T

M. Barnsley introduced the concept of fractal interpolation function as an alternative method for construction of interpolation functions. Such a function is the attractor of an iterated function system comprising Banach contractions (on the product of two real compact intervals) with respect to the second variable. In the present paper we enlarge classical Barnsley's framework by allowing the constitutive functions of the system to be Edelstein contractions in the second variable. As the class of Edelstein contractions contains the class of Matkowski contractions, the class of Meir-Keeler contractions, the class of F-contractions and the class of θ-contractions, we provide a much more flexible framework for the construction of fractal interpolation functions. We also obtain a result concerning the estimation of lower and upper box dimensions of the graph of a fractal interpolation function constructed via our general method.

1. Introduction

An iterated function system (for short IFS), a concept due to J. Hutchinson (see [16]), consists of a finite family of Banach contractions on a complete metric space. It plays a pivotal role in the construction of a class of fractal sets (called attractors of IFSs or Hutchinson-Barnsley fractals) having plentiful applications. This prompts the question if Hutchinson's theory is valid under weaker contraction conditions. Several positive answers have been provided for the following larger classes of contractions: Matkowski contractions (see [15]), Meir-Keeler contractions (see [11]), F-contractions (see [35]), convex contractions (see [24]), θ-contractions (see [28]), Hardy-Rogers contractions (see [13])) and Edelstein contractions (see [25]). In this way, the overwhelming role of various generalizations of Banach-Caccioppoli-Picard contraction principle in the theory of IFSs was highlighted.

M. Barnsley (see [1]) introduced the concept of fractal interpolation function (for short FIF) as an alternative method for construction of interpolation functions. Such a FIF is the attractor of an IFS $\mathcal{S} =$

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$\big((I \times \mathcal{K}, \|.\|_2), (f_i)_{i\in\{1,2,\dots,N\}}\big)$, where:
- I and $\mathcal{K}$ are real compact intervals;
- there exist $l_i : I \to I$ and $g_i : I \times \mathcal{K} \to \mathcal{K}$ such that:
- $f_i : I \times \mathcal{K} \to I \times \mathcal{K}$ is given by

$$f_i(x, y) = (l_i(x), g_i(x, y)),$$

for every $(x, y) \in I \times \mathcal{K}$ and $i \in \{1, 2, \dots, N\}$;
- g_i is a Banach contraction with respect to the second variable y for every $i \in \{1, 2, \dots, N\}$.

Let us mention some of the major advantages of FIFs theory:
- the flexibility to generate either smooth or non-smooth approximants, depending on the problem that is to be studied;
- the capability to encapsulate the self-similar and fracture nature of complex aspects of the real world;
- the possibility to measure the complexity of the phenomenon being studied via the Hausdorff and Minkowski dimensions of FIF's graph.

Overall, the iterative approach used in the FIFs theory allows a more versatile and adaptive approximation. This makes this kind of functions a powerful tool in various fields as image processing (see [4]), bio-engineering (see [10]), financial series (see [19]), signal processing (see [37]), etc.

FIFs have been generalized in multiple ways:
(i) recurrent fractal interpolation functions, which are giving partial self-similarity interpolation function (see [2,6]);
(ii) hidden-variable fractal interpolation functions (see [3]);
(iii) Hermit or spline fractal interpolation functions (see [7,8]);
(iv) α-fractal functions (see [26,27]);
(v) histopolating fractal functions, which excluded the two properties inherent in fractal interpolation, namely continuity and interpolation (see [5]);
(vi) multi-variable fractal interpolation functions, which are giving the self-similar interpolation function in higher dimensional space (see [21]);
(vii) fractal interpolation functions for a countable data set (see [34]);
(viii) generalized non-linear fractal interpolation functions by using weak contractions namely Matkowski contractions and Kannan contractions (see [9,30,31,33]);

We emphasize that classical Barnsley's framework for construction of FIFs has been enlarged by S. Ri who was able, via Rakotch and Matkowski fixed point theorems, to weaken the assumptions on the functions g_i.

In the present paper we enlarge Ri's framework by allowing the functions g_i to be Edelstein contraction in the second variable (see Remark 4.2). As the class of Edelstein contractions contains the class of Matkowski contractions, the class of Meir-Keeler contractions, the class of F-contractions and the class of θ-contractions, we provide a much more flexible framework for the construction of fractal interpolation functions. We also come up with a result concerning the estimation of the lower and upper box dimensions of the graph of a FIF constructed via our general method (see Remark 5.1).

2. Notation and terminology

$\mathbb{N}$ denotes the set $\{1, 2, \dots\}$.
If $f : A \to B$ is a function, then G_f designates the graph of f.
For a function $f : X \to X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we mean $\underbrace{f \circ \cdots \circ f}_{n\text{-times}}$.
For a function $f : X \to X$, where (X, d) is a metric space, by $l(f)$ we mean the lower Lipschitz constant of f and by $L(f)$ we mean the upper Lipschitz constant of f. So

$$l(f) = \inf_{\substack{x,y\in X\\ x\neq y}} \frac{d(f(x),f(y))}{d(x,y)} \qquad \text{and} \qquad L(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{d(f(x),f(y))}{d(x,y)}.$$

For a nonempty bounded subset A of $\mathbb{R}^2$ by $\underline{\dim}_B(A)$ we mean the lower box dimension of A and by $\overline{\dim}_B(A)$ we mean the upper box dimension of A. So

$$\underline{\dim}_B(A) = \liminf_{\substack{\delta\to 0\\ \delta>0}} \frac{\log N_\delta(A)}{\log(1/\delta)} \qquad \text{and} \qquad \overline{\dim}_B(A) = \limsup_{\substack{\delta\to 0\\ \delta>0}} \frac{\log N_\delta(A)}{\log(1/\delta)},$$

where $N_\delta(A)$ represents the minimum number of boxes with side length δ and sides parallel to the axes, whose union contains A. If $\underline{\dim}_B(A) = \overline{\dim}_B(A)$, this common value is denoted by $\dim_B(A)$ and it is called the box dimension of A.

If (X,d) is a metric space:

- by $P_{\mathrm{cp}}(X)$ we denote the set $\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\}$;
- by h we mean the Hausdorff-Pompeiu metric;
- an operator $f : X \to X$ is called Picard if it has a unique fixed point x^* and $\lim_{n\to\infty} f^{[n]}(x) = x^*$ for every $x \in X$;
- a function $f : X \to X$ is called Banach contraction if there exists a constant $C \in [0,1)$ such that

$$d(f(x),f(y)) \leq C\ d(x,y),$$

for every $x, y \in X$;

- a function $f : X \to X$ is called Matkowski contraction if

$$d(f(x),f(y)) \ \leq \varphi(d(x,y)),$$

for every $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a non-decreasing map satisfying $\lim_{n\to\infty} \varphi^{[n]}(t) = 0$ for any $t > 0$;

- a function $f : X \to X$ is called Meir-Keeler contraction if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ the following implication is valid:

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \ \Rightarrow\ d(f(x),f(y)) < \varepsilon;$$

- a function $f : X \to X$ is called F-contraction if for every $x, y \in X$ the following implication is valid:

$$Tx \neq Ty \ \Rightarrow\ \tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

where $\tau > 0$ and $F : (0,\infty) \to \mathbb{R}$ satisfies the following conditions: (i) it is strictly increasing; (ii) for every sequence $(t_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ the following equivalence is true: $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$; (iii) there exists $k \in (0,1)$ satisfying $\lim_{\substack{t\to 0\\ t>0}} t^k F(t) = 0$;

- a function $f : X \to X$ is called θ-contraction if

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k,$$

for every $x, y \in X$, where $k \in [0,1)$ and $\theta : [0,\infty) \to [1,\infty)$ satisfies the following conditions: (i) it is non-decreasing; (ii) for each sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ the following equivalence is true: $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$; (iii) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{\substack{t\to 0\\ t>0}} \frac{\theta(t)-1}{t^r} = l$;

- a function $f : X \to X$ is called φ-contraction if

$$d(f(x), f(y)) \leq \varphi(d(x, y)),$$

for every $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ satisfies the condition $\varphi(t) < t$ for all $t > 0$;
- a function $f : X \to X$ is called Edelstein contraction (or contractive) if for every $x, y \in X$ the following implication is valid:

$$x \neq y \;\Rightarrow\; d(f(x), f(y)) < d(x, y).$$

3. Preliminary facts

Theorem 3.1. *Let (X, d) be a complete metric space and $f : X \to X$. Then f is a Picard operator provided that it is:*

(i) Matkowski contraction;
(ii) Meir-Keeler contraction;
(iii) F-contraction;
(iv) θ-contraction.

Remark 3.1. For (i) see [22], for (ii) see [23], for (iii) see [36] and for (iv) see [18].

Theorem 3.2. *(see [12]) Let (X, d) be a compact metric space and $f : X \to X$. Then f is a Picard operator provided that it is an Edelstein contraction.*

Remark 3.2.

(i) Each Banach contraction is Matkowski contraction, F-contraction, Meir-Keeler contraction and θ-contraction.
(ii) The Matkowski contractions, F-contractions, Meir-Keeler contractions, θ-contractions and φ-contractions are Edelstein contractions.
(iii) There is no connection between the notions of Matkowski contraction, Meir-Keeler contraction, F-contraction and θ-contraction.
(iv) If $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $\lim_{n\to\infty} \varphi^{[n]}(t) = 0$ for any $t > 0$, then $\varphi(t) < t$ for all $t > 0$. In particular, each Matkowski contraction is φ-contraction.
(v) The collection of Edelstein contractions contains lot of the Picard operators.
(vi) If f is an Edelstein contraction on a compact metric space, then it is a Meir-Keeler contraction (see [23]). So, on compact spaces, the class of Meir-Keeler contractions coincide with the class of Edelstein contractions. This remark is crucial for our paper.

For details concerning the relation between the above mentioned type of contractions see [17] and [29]. The following two examples will be used in Sections 4 and 5.

Example 3.1. Let $f : [-1, \infty] \to [-1, \infty]$ be defined by

$$f(x) = \frac{1 + x}{2 + x},$$

for every $x \in [-1, \infty)$.
Since

$$|f(x)-f(y)| = \left|\frac{x-y}{1+(1+x)+(1+y)+(1+x)(1+y)}\right| \le \frac{|x-y|}{1+|x-y|} = \varphi(|x-y|),$$

for every $x, y \in [-1, \infty)$, where φ is given by

$$\varphi(t) = \frac{t}{1+t},$$

for every $t \ge 0$, f is a Matkowski contraction.
Note that f is not a Banach contraction.
Indeed, if this is not the case, there exists $C \in [0, 1)$ such that

$$|f(x)-f(y)| \le C|x-y|,$$

for all $x, y \in [-1, \infty)$. In particular choosing $x = -1$ and $y = -1 + \frac{1}{n}$ with $n \in \mathbb{N}$, we get

$$\frac{n}{1+n} \le C,$$

for all $n \in \mathbb{N}$. By passing to limit as n goes to ∞, we obtain the contradiction $1 \le C$.

Example 3.2. Let $f : [-1, 1] \to [-1, 1]$ be defined by

$$f(x) = \frac{1}{2}x^2,$$

for every $x \in [-1, 1]$.
Since

$$|f(x)-f(y)| = \frac{1}{2}|x^2-y^2| = \frac{1}{2}|x+y||x-y|,$$

for every $x, y \in [-1, 1]$, f is an Edelstein contraction as for $x, y \in [-1, 1]$ with $x \ne y$ we have $|x+y| < 2$.
Note that f is not a Banach contraction.
Indeed, if this is not the case, there exists $C \in [0, 1)$ such that

$$|f(x)-f(y)| \le C|x-y|,$$

for all $x, y \in [-1, 1]$. In particular choosing $x = 1$ and $y = 1 - \frac{1}{n}$ with $n \in \mathbb{N}$, we get

$$\frac{2n-1}{2n} \le C,$$

for all $n \in \mathbb{N}$. Hence by passing to limit as n goes to ∞, we obtain the contradiction $1 \le C$.

Definition 3.1. A pair $\big((X, d), (f_i)_{i\in\{1,2,\ldots,N\}}\big)$, where (X, d) is a complete metric space and $f_i : X \to X$ is continuous for each $i \in \{1, 2, \ldots, N\}$, is called an Iterated Function System (for short IFS) and it is denoted by $\mathcal{S}$.
The function $F_{\mathcal{S}} : P_{\text{cp}}(X) \to P_{\text{cp}}(X)$, given by

$$F_{\mathcal{S}}(K) = \bigcup_{i=1}^{N} f_i(K),$$

for every $K \in P_{\text{cp}}(X)$, is called the fractal operator associated with $\mathcal{S}$.
If $F_{\mathcal{S}}$ is a Picard operator, then its fixed point $A_{\mathcal{S}}$ is called the attractor of $\mathcal{S}$.

Theorem 3.3. *Let* $\mathcal{S} = \left((X,d),(f_i)_{i\in\{1,2,\dots,N\}}\right)$ *be an IFS. Then* $F_{\mathcal{S}}$ *is a Picard operator provided that the functions* f_i *are:*

- *(i) Banach contractions;*
- *(ii) Matkowski contractions;*
- *(iii) Meir-Keeler contractions;*
- *(iv)* F*-contractions;*
- *(v)* θ*-contractions;*
- *(vi) Edelstein contractions and* (X,d) *is compact.*

Remark 3.3. For (i) see [16], for (ii) see [15], for (iii) see [11], for (iv) see [35], for (v) see [28] and for (vi) see [25].

Following the framework adopted in [1], let us consider $N\in\mathbb{N}$ and a set of data points $\{(x_i,y_i)\in\mathbb{R}^2 : i\in\{0,1,\dots,N\}\}$ such that $x_0<x_1<\dots<x_N$. Let $I:=[x_0,x_N], I_i:=[x_{i-1},x_i]$ and $l_i : I\to I_i$ be given by

$$l_i(x) = a_i x + b_i,$$

for every $x\in I$, where $a_i=\frac{x_i-x_{i-1}}{x_N-x_0}$ and $b_i=x_{i-1}-\frac{x_i-x_{i-1}}{x_N-x_0}x_0$ for each $i\in\{0,1,\dots,N\}$. Note that

$$l_i(x_0)=x_{i-1} \quad \text{and} \quad l_i(x_N)=x_i,$$

for all $i\in\{1,2,\dots,N\}$. Further, we consider the continuous functions $g_i : I\times\mathcal{K}\to\mathcal{K}$ which are Lipschitz with respect to the first variable and which satisfy

$$g_i(x_0,y_0)=y_{i-1} \quad \text{and} \quad g_i(x_N,y_N)=y_i,$$

where $\{y_0,y_1,\dots,y_N\}\subset\mathcal{K}\in P_{\mathrm{cp}}(\mathbb{R})$.
We also consider the IFS

$$\mathcal{S}=\left((I\times\mathcal{K},\|.\|_2),(f_i)_{i\in\{1,2,\dots,N\}}\right),$$

where $f_i : I\times\mathcal{K}\to I\times\mathcal{K}$ is defined by

$$f_i(x,y)=(l_i(x),g_i(x,y)),$$

for every $(x,y)\in I\times\mathcal{K}$ and $i\in\{1,2,\dots,N\}$.
The case when g_i's are Banach contractions with respect to the second variable is treated in [1].
It turns out that in this case f_i's are Banach contractions with respect to a certain metric d, which is equivalent to the Euclidean metric. Hence the IFS $\mathcal{S}=\left((I\times\mathcal{K},d),(f_i)_{i\in\{1,2,\dots,N\}}\right)$ has a unique attractor $A_{\mathcal{S}}$.
Moreover, M. Barnsley proved that there exists a unique continuous function $f^* : I\to\mathcal{K}$ such that

$$G_{f^*}=A_{\mathcal{S}}$$

and

$$f^*(x_i)=y_i,$$

for all $i \in \{0, 1, \ldots, N\}$.
The functions obtained in this way are called fractal interpolation functions (for short FIFs).
If

$$g_i(x, y) = c_i x + \alpha_i y + d_i,$$

for every $i \in \{1, 2, \ldots, N\}$ and $(x, y) \in I \times \mathcal{K}$, where c_i, α_i, d_i are constants and $|\alpha_i| < 1$, then we get the so called affine FIFs.
Later on, in [30], S. Ri proposed a larger framework in order to get fractal interpolation functions, via Matkowski contractions. More precisely he considered the case when g_i's are Matkowski contractions with respect to the second variable.

4. Contractive fractal interpolation function

In this section, we provide a more general framework for getting fractal interpolation functions by using Edelstein contractions.

Let $N \in \mathbb{N}$ and $\{(x_i, y_i) \in \mathbb{R}^2 : i \in \{0, 1, \ldots, N\}\}$ be a set of data set such that $x_0 < x_1 < \cdots < x_N$. Let $l_i : I \to I_i$ be given by

$$l_i(x) = a_i x + b_i,$$

for every $x \in I$, where $I := [x_0, x_N], I_i := [x_{i-1}, x_i], a_i = \frac{x_i - x_{i-1}}{x_N - x_0}$ and $b_i = x_{i-1} - \frac{x_i - x_{i-1}}{x_N - x_0} x_0$ for each $i \in \{1, 2, \ldots, N\}$.
Let us also consider $g_i : I \times \mathcal{K} \to \mathcal{K}$ Lipschitz with respect to the first variable, Edelstein contraction with respect to the second variable and satisfying

$$g_i(x_0, y_0) = y_{i-1} \quad \text{and} \quad g_i(x_N, y_N) = y_i, \tag{4.1}$$

for every $i \in \{1, 2, \ldots, N\}$, where $\{y_0, y_1, \ldots, y_N\} \subset \mathcal{K} \in P_{\text{cp}}(\mathbb{R})$.
Thus, there exist $r_i \geq 0$ and an Edelstein contraction $h_i : \mathcal{K} \to \mathcal{K}$ such that

$$|g_i(x, y) - g_i(x', y')| \leq r_i |x - x'| + |h_i(y) - h_i(y')|, \tag{4.2}$$

for every $i \in \{1, 2, \ldots, N\}$ and $(x, y), (x', y') \in I \times \mathcal{K}$.
Let us consider the IFS $\mathcal{S} = \big((I \times \mathcal{K}, \|.\|_2), (f_i)_{i \in \{1,2,\ldots,N\}}\big)$, where $f_i : I \times \mathcal{K} \to I \times \mathcal{K}$ is defined by

$$f_i(x, y) = (l_i(x), g_i(x, y)),$$

for every $(x, y) \in I \times \mathcal{K}$ and $i \in \{1, 2, \ldots, N\}$.
Let us also consider the complete metric space

$$\mathcal{C}(I) := \{f : I \to \mathcal{K} : f \text{ is continuous}, f(x_0) = y_0 \text{ and } f(x_N) = y_N\}$$

endowed with the uniform metric ρ and

$$\mathcal{C}^*(I) := \{f : I \to \mathcal{K} : f \text{ is continuous and } f(x_i) = y_i \text{ for every } i \in \{0, 1, \ldots, N\}\}.$$

Let $T : \mathcal{C}(I) \to \mathcal{C}(I)$ given by

$$T(f)(x) = g_i(l_i^{-1}(x), f(l_i^{-1}(x))),$$

for all $x \in I_i$ and $i \in \{1, 2, \dots, N\}$.
Note that T is well defined.

Lemma 4.1.

$$T(f) \in \mathcal{C}^*(I),$$

for all $f \in \mathcal{C}(I)$.

Proof. For $f \in \mathcal{C}(I)$ and $i \in \{1, 2, \dots, N-1\}$, taking into account the definition of T, viewing x_i as an element of $[x_{i-1}, x_i]$, we have

$$T(f)(x_i) = g_i(l_i^{-1}(x_i), f(l_i^{-1}(x_i))) = g_i(x_N, f(x_N)) = g_i(x_N, y_N) = y_i$$

and, viewing x_i as an element of $[x_i, x_{i+1}]$, we have

$$T(f)(x_i) = g_{i+1}(l_{i+1}^{-1}(x_i), f(l_{i+1}^{-1}(x_i))) = g_{i+1}(x_0, f(x_0)) = g_{i+1}(x_0, y_0) = y_i.$$

Similarly, we can get $T(f)(x_0) = y_0$ and $T(f)(x_N) = y_N$.
The continuity of $T(f)$ is ensured by the continuity of the functions g_i, l_i^{-1} and f. $\square$

Theorem 4.1. *T is a Meir-Keeler contraction.*

Proof. Let us consider $\varepsilon > 0$ arbitrarily chosen, but fixed.
In view of Remark 3.2 (vi), there exists $\delta_i > 0$ such that, for every $x \in I, y, y' \in \mathcal{K}$ and $i \in \{1, 2, \dots, N\}$, the following implication is valid:

$$\varepsilon \leq |y - y'| < \varepsilon + \delta_i \quad \Rightarrow \quad |g_i(x, y) - g_i(x, y')| < \varepsilon. \tag{4.3}$$

Let us choose

$$\delta = \min\{\delta_i \ : \ i \in \{1, 2, \dots, N\}\}.$$

Claim: $\rho(Tf, Tg) < \varepsilon$ provided that $f, g \in \mathcal{C}(I)$ are such that $\varepsilon \leq \rho(f, g) < \varepsilon + \delta$.
Justification of the claim. Note that $f(x), g(x) \in \mathcal{K}$ and

$$|f(x) - g(x)| \ \leq \ \rho(f, g) \ < \ \varepsilon + \delta \ \leq \ \varepsilon + \delta_i, \tag{4.4}$$

for every $x \in I$ and $i \in \{1, 2, \dots, N\}$.
We have

$$|g_i(x, f(x)) - g_i(x, g(x))| < \varepsilon, \tag{4.5}$$

for every $x \in I$ and $i \in \{1, 2, \dots, N\}$.
Indeed, if $\varepsilon \leq |f(x) - g(x)|$, then we get the conclusion via (4.3) and (4.4). Otherwise, via (4.2), we obtain $|g_i(x, f(x)) - g_i(x, g(x))| \leq |f(x) - g(x)| < \varepsilon$.
As I is compact, the continuity of f and g_i's, via (4.5), ensures the validity of the following relation:

$$\max_{x \in I} |g_i(x, f(x)) - g_i(x, g(x))| < \varepsilon, \tag{4.6}$$

for every $i \in \{1, 2, \ldots, N\}$.
Consequently

$$\begin{aligned}\rho(T(f), T(g)) &= \max_{x\in I} |T(f)(x) - T(g)(x)| \\ &= \max_{i\in\{1,2,\ldots,N\}} \max_{x\in I_i} |g_i(l_i^{-1}(x), f(l_i^{-1}(x))) - g_i(l_i^{-1}(x), g(l_i^{-1}(x)))| \\ &= \max_{i\in\{1,2,\ldots,N\}} \max_{x\in I} |g_i(x, f(x)) - g_i(x, g(x))| \overset{(4.6)}{<} \varepsilon,\end{aligned}$$

so the justification of the claim is finished and the proof is complete. □

Corollary 4.1. *T is a Picard operator.*

Proof. The conclusion results from Theorem 3.1 (*ii*) and from Theorem 4.1. □

Remark 4.1. Corollary 4.1 ensures the existence of a unique $f^* \in \mathcal{C}(I)$ such that

$$T(f^*) = f^* \quad \text{and} \quad \lim_{n\to\infty} T^{[n]}(f) = f^*,$$

for all $f \in \mathcal{C}(I)$.

As, generally, we do not know that $\mathcal{S} = \big((I \times \mathcal{K}, \|.\|_2), (f_i)_{i\in\{1,2,\ldots,N\}}\big)$ has attractor, in order to overcome this shortcoming, we will introduce a metric equivalent with the Euclidean one.

Remark 4.1. If $\theta > 0$, then $d_\theta : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$, given by

$$d_\theta((x, y), (x', y')) := |x - x'| + \theta|y - y'|,$$

for every $(x, y), (x', y') \in \mathbb{R}^2$, is a metric on $\mathbb{R}^2$ which is equivalent with $\|.\|_2$.

Theorem 4.2. *There exists $\theta > 0$ such that each f_i is an Edelstein contraction with respect to d_θ for every $i \in \{1, 2, \ldots, N\}$.*

Proof. Let us choose

$$\theta = \frac{1 - \max\limits_{i\in\{1,2,\ldots,N\}} a_i}{1 + \max\limits_{i\in\{1,2,\ldots,N\}} r_i}.$$

Note that $\theta > 0$, since $a_i \in (0, 1)$ for all $i \in \{1, 2, \ldots, N\}$.
Then,

$$\begin{aligned}d_\theta(f_i(x, y), f_i(x', y')) &= |l_i(x) - l_i(x')| + \theta|g_i(x, y) - g_i(x', y')| \\ &\overset{(4.2)}{\leq} a_i|x - x'| + \theta(r_i|x - x'| + |h_i(y) - h_i(y')|) \\ &\leq (a_i + \theta r_i)|x - x'| + \theta|h_i(y) - h_i(y')|,\end{aligned}$$

for every $(x, y), (x', y') \in I \times \mathcal{K}$ and $i \in \{1, 2, \ldots, N\}$.
Since $a_i + \theta r_i < 1$ and h_i is an Edelstein contraction, we get

$$d_\theta(f_i(x,y), f_i(x',y')) < |x - x'| + \theta|y - y'| = d_\theta((x,y),(x',y')),$$

for every $(x,y),(x',y') \in I \times \mathcal{K}$ with $(x,y) \neq (x',y')$ and $i \in \{1,2,\ldots,N\}$. $\square$

Corollary 4.2. *$F_{\mathcal{S}}$ is a Picard operator.*

Proof. Since $\|.\|_2$ and d_θ are equivalent metrics, $(I \times \mathcal{K}, d_\theta)$ is compact and the corresponding Hausdorff metrics are also equivalent, via Theorems 3.3, vi) and Theorems 4.2, we conclude that $F_{\mathcal{S}}$ is a Picard operator. $\square$

Corollary 4.3. *G_{f^*} is the attractor of $\mathcal{S}$.*

Proof. As

$$\begin{aligned} f_i(G_{f^*}) &= \{f_i(x, f^*(x)) : x \in I\} = \{(l_i(x), g_i(x, f^*(x))) : x \in I\} \\ &\overset{\text{T(f*)=f*}}{=} \{(l_i(x), f^*(l_i(x))) : x \in I\} = \{(x, f^*(x)) : x \in I_i\}, \end{aligned}$$

for every $i \in \{1,2,\ldots,N\}$, we get

$$G_{f^*} = \bigcup_{i=1}^{N} \{(x, f^*(x)) : x \in I_i\} = \bigcup_{i=1}^{N} f_i(G_{f^*}) = F_{\mathcal{S}}(G_{f^*}).$$

Since $G_{f^*} \in P_{\text{cp}}(I \times \mathcal{K})$, the uniqueness of the fixed point of $F_{\mathcal{S}}$ ensures that

$$G_{f^*} = A_{\mathcal{S}}. \quad \square$$

Remark 4.2. Since $f^* = T(f^*) \overset{\text{Lemma 4.1}}{\in} \mathcal{C}^*(I)$, we conclude that

$$f^*(x_i) = y_i,$$

for every $i \in \{0,1,\ldots,N\}$. As $G_{f^*} = A_{\mathcal{S}}$ and the constitutive functions of $\mathcal{S}$ are Edelstein contractions with respect to the second variable, we call f^* contractive fractal interpolation function.

Remark 4.2. One can easily check that

$$G_{T^{[n]}(f)} = F_{\mathcal{S}}^{[n]}(G_f),$$

for all $n \in \mathbb{N}$ and $f \in \mathcal{C}(I)$.
Consequently

$$\lim_{n\to\infty} G_{T^{[n]}(f)} = A_{\mathcal{S}} = G_{f^*},$$

for all $f \in \mathcal{C}(I)$.

Example 4.1. We shall present a concrete example of our interpolation framework involving g_i's which are Edelstein contractions with respect to the second variable.
Specifically, we take $N = 3$ and

$$x_0 = 0, x_1 = 1/2, x_2 = 3/4, x_3 = 1, y_0 = 1/4, y_1 = 1/3, y_2 = 1/2, y_3 = 1.$$

Let us consider $g_i : [0,1] \times [-1,1] \to [-1,1]$ given by

$$g_i(x,y) = c_i x + h(y) + d_i,$$

for every $(x,y) \in [0,1] \times [-1,1]$ and $i \in \{1,2,3\}$, where $h : [-1,1] \to [-1,1]$ is given by

$$h(y) = \frac{1}{2}y^2,$$

for every $y \in [-1,1]$.
Therefore, we get

$$\begin{aligned} l_1(x) &= \frac{1}{2}x, & g_1(x,y) &= \frac{-37}{96}x + \frac{1}{2}y^2 + \frac{7}{32}, \\ l_2(x) &= \frac{1}{4}x + \frac{1}{2}, & g_2(x,y) &= \frac{-29}{96}x + \frac{1}{2}y^2 + \frac{29}{96}, \\ l_3(x) &= \frac{1}{4}x + \frac{3}{4}, & g_3(x,y) &= \frac{1}{32}x + \frac{1}{2}y^2 + \frac{15}{32}, \end{aligned}$$

for all $x \in [0,1]$ and $y \in [-1,1]$.
Note that g_i's are Lipschitz with respect to x and (in view of Example 3.2) Edelstein contractions, but not Banach contractions, with respect to y.

Open question: Since there exist iterated function systems $\big((X,d),(f_i)_{i\in\{1,2,\dots,N\}}\big)$ having attractor, but with $\max\{L(f_i) : i \in \{1,2,\dots,N\}\} > 1$ (see Example 3.2 from [20]), the following question naturally arises: Is it possible to obtain fractal interpolation functions in a framework involving g_i's which are not Edelstein contractions with respect to the second variable?

5. Box dimension of a contractive FIF

This section, which is devoted to the box dimension of contractive interpolation functions, draws inspiration from [32]. As the main arguments are similar with the ones in the above mentioned paper, we shall present just the main steps of the proof, skipping the details.
We consider the particular case of the data set $\{(x_i,y_i) \in \mathbb{R}^2 \ : \ x_i = \frac{i}{N}, i \in \{0,1,\dots,N\}\}$, $I = [0,1]$ and $g_i : [0,1] \times \mathcal{K} \to \mathcal{K}$ defined by

$$g_i(x,y) = c_i x + h_i(y) + d_i,$$

for every $x \in [0,1], y \in \mathcal{K}$ and $i \in \{1,2,\dots,N\}$, where c_i and d_i are constants and $h_i : \mathcal{K} \to \mathcal{K}$ is an Edelstein contraction for every $i \in \{1,2,\dots,N\}$.
Hence, taking into account the definition of l_i and (4.1), we have

$$a_i = \frac{1}{N}, \quad b_i = \frac{i-1}{N}, \quad c_i = (y_i - y_{i-1}) - (h_i(y_N) - h_i(y_0)) \text{ and } d_i = y_{i-1} - h_i(y_0),$$

for every $i \in \{1,2,\dots,N\}$.
Let us denote by G the graph of the corresponding contractive fractal interpolation function f^*.
Since

$$N_{\frac{1}{N^r}}(G) \leq N_r^*(G) \leq 2N_{\frac{1}{N^r}}(G),$$

we have

$$\underline{\dim}_B(G) = \liminf_{r\to\infty} \frac{\log N_r^*(G)}{\log(N^r)} \qquad \text{and} \qquad \overline{\dim}_B(G) = \limsup_{r\to\infty} \frac{\log N_r^*(G)}{\log(N^r)}, \tag{5.7}$$

where $N_r^*(G)$ represents the minimum number of boxes belonging to the family

$$\mathcal{C} := \left\{ \left[\frac{k-1}{N^r}, \frac{k}{N^r}\right] \times \left[z, z + \frac{1}{N^r}\right] : r \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}, z \in \mathbb{R} \right\}$$

needed to cover G.
If $C_r(G) \subset \mathcal{C}$ is a cover of G consisting of $N_r^*(G)$ boxes and $k \in \{1, 2, \ldots, N^r\}$, where $r \in \mathbb{N} \cup \{0\}$, we consider:

- $C_r^k(G)$ the collection of all squares in $C_r(G)$ whose x-axis lie between $\frac{k-1}{N^r}$ and $\frac{k}{N^r}$
- $N_r^k(G)$ the number of elements in $C_r^k(G)$
- $R_r^k(G)$ a rectangle of width $\frac{1}{N^r}$ and height $\frac{N_r^k(G)}{N^r}$ which consists of squares in $C_r^k(G)$.

Note that

$$N_r^*(G) = \sum_{k=1}^{N^r} N_r^k(G).$$

Theorem 5.1. *In the above framework, we have*

$$\overline{\dim}_B(G) \leq 1 + \log_N(\gamma),$$

provided that

$$\gamma := \sum_{i=1}^{N} L(h_i) > 1.$$

Moreover if there exists $i \in \{1, 2, \ldots, N\}$ such that h_i is a Banach contraction, then

$$\overline{\dim}_B(G) \leq 1 + \log_N(\gamma) < 2.$$

Proof. For $r \in \mathbb{N},\ i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, N^r\}$, let us consider

$$l(k, i) = N^{r+1} l_i \left(\frac{k}{N^r}\right) = k + N^r(i-1).$$

Note that

$$f_i(R_r^k(G)) \subset \left[\frac{l(k,i)-1}{N^{r+1}}, \frac{l(k,i)}{N^{r+1}}\right] \times \mathbb{R}$$

and that it is contained in a rectangle of height $\frac{L(h_i)N_r^k(G)}{N^r} + \frac{|c_i|}{N^r}$ for every $k \in \{1, 2, \ldots, N^r\}$ and $i \in \{1, 2, \ldots, N\}$.
Since

$$G = \bigcup_{i=1}^{N} f_i(G) \subset \bigcup_{i=1}^{N} f_i\left(\bigcup_{k=1}^{N^r} R_r^k(G)\right) = \bigcup_{i=1}^{N} \bigcup_{k=1}^{N^r} f_i(R_r^k(G)),$$

we get

$$N^*_{r+1}(G) = \sum_{i=1}^{N}\sum_{k=1}^{N^r} N^{l(k,i)}_{r+1}(G) \leq \sum_{i=1}^{N}\sum_{k=1}^{N^r}\left(\left(\frac{L(h_i)N^k_r(G)}{N^r} + \frac{|c_i|}{N^r}\Big/\frac{1}{N^{r+1}}\right)+1\right)$$
$$= N\gamma N^*_r(G) + N^{r+1}c',$$

where $c' = \sum_{i=1}^{N}|c_i| + 1$.
Via the mathematical induction method, we get

$$N^*_r(G) \leq (N\gamma)^r c'',$$

for every $r \in \mathbb{N}$, where $c'' = \left(N^*_0(G) + \frac{c'}{\gamma-1}\right)$.
Therefore taking into account (5.7), we have

$$\overline{\dim}_B(G) \leq \limsup_{r\to\infty}\frac{\log((N\gamma)^r c'')}{\log(N^r)}$$
$$= \limsup_{r\to\infty}\frac{\log(N^r)+\log(\gamma^r)+\log(c'')}{\log(N^r)} = 1 + \log_N(\gamma).$$

If there exists $i \in \{1, 2, \dots, N\}$ such that h_i is a Banach contraction, then $\gamma < N$, so

$$\overline{\dim}_B(G) \leq 1 + \log_N(\gamma) < 2. \quad \square$$

Lemma 5.1. *In the previous framework, assume that the interpolation points are not collinear, i.e. there exists* $i \in \{1, 2, \dots, N\}$ *such that* $V_i := y_i - (y_0 + \frac{i}{N}(y_N - y_0)) \neq 0$. *If*

(i) $V_j < 0$ *for some* $j \in \{1, 2, \dots, N\}$ *and* $\alpha_1 := \sum_{i=1}^{N} l_1(h_i) > 1$, *where*

$$l_1(h_i) = \begin{cases} l(h_i), & \text{if } h_i \text{ is increasing and convex} \\ 0, & \text{otherwise} \end{cases}$$

or

(ii) $V_j > 0$ *for some* $j \in \{1, 2, \dots, N\}$ *and* $\alpha_2 := \sum_{i=1}^{N} l_2(h_i) > 1$, *where*

$$l_2(h_i) = \begin{cases} l(h_i), & \text{if } h_i \text{ is increasing and concave} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\lim_{r\to\infty}\frac{N^*_r(G)}{N^r} = \infty.$$

Proof. Note that, according to [14], we have

$$|V_j| \leq \max\{|y_j - y_0|, |y_j - y_N|\},$$

for every $j \in \{1, 2, \dots, N\}$.
Since G is the graph of a continuous function passing through $(0, y_0)$, $(\frac{j}{N}, y_j)$ and $(1, y_N)$, we have

$$N_r^*(G) \geq [|V_j|N^r],$$

for every $j \in \{1, 2, \dots, N\}$ and $r \in \mathbb{N}$, where $[.]$ denotes the integer part of a real number.
Let us suppose that *i*) is valid and let $j \in \{1, 2, \dots, N\}$ be such that $V_j < 0$.
For $i \in \{1, 2, \dots, N\}$ such that h_i is increasing and convex, we have

$$\begin{aligned} g_i(\frac{j}{N}, y_j) &\leq g_i(\frac{j}{N}, y_0 + \frac{j}{N}(y_N - y_0)) \\ &= h_i((1-\frac{j}{N})y_0 + \frac{j}{N}y_N) - ((1-\frac{j}{N})h_i(y_0) + \frac{j}{N}h_i(y_N)) + \frac{j}{N}(y_i - y_{i-1}) + y_{i-1} \\ &\leq \frac{j}{N}(y_i - y_{i-1}) + y_{i-1} \leq \max\{y_i, y_{i-1}\} = \max\{g_i(1, y_N), g_i(0, y_0)\}, \end{aligned}$$

and since $f_i(0, y_0), f_i(\frac{j}{N}, y_j)$ and $f_i(1, y_N)$ belong to G, we get

$$\sum_{k=N^r l_i(0)+1}^{N^r l_i(1)} N_r^k(G) \geq \left[\frac{g_i(\frac{j}{N}, y_0 + \frac{j}{N}(y_N - y_0)) - g_i(\frac{j}{N}, y_j)}{\frac{1}{N^r}} \right] \geq [l(h_i)\ |V_j|N^r],$$

for every $r \in \mathbb{N}$.
Thus

$$N_r^*(G) = \sum_{i=1}^{N} \sum_{k=N^r l_i(0)+1}^{N^r l_i(1)} N_r^k(G)\ \geq\ \sum_{i=1}^{N} [l_1(h_i)|V_j|N^r],$$

and using the mathematical induction method, we get

$$N_r^*(G) \geq \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_r=1}^{N} [l_1(h_{i_1}) l_1(h_{i_2}) \dots l_1(h_{i_r}) |V_j| N^r] \geq ((\alpha_1)^r |V_j| - 1)N^r,$$

for every $r \in \mathbb{N}$.
Hence

$$\lim_{r \to \infty} \frac{N_r^*(G)}{N^r} = \infty.$$

A similar argument works if *ii*) is valid. □

Theorem 5.2. *In the framework of Lemma 5.1, we have*

$$1 < 1 + \log_N(\alpha) \leq \underline{\dim}_B(G),$$

where $\alpha := \sum_{i=1}^{N} l(h_i)$.

Proof. Note that for $A \in C_r^k(G)$, where $r \in \mathbb{N} \cup \{0\}$ and $k \in \{1, 2, \dots, N^r\}$, we have

$$f_i(A) \cap G \neq \emptyset,$$

for all $i \in \{1, 2, \dots, N\}$, since $A \cap G \neq \emptyset$.
Therefore, $C_{r+1}^{l(k,i)}(G)$ must cover at least a rectangle of height $\frac{l(h_i)(N_r^k(G)-2)}{N^r} - \frac{|c_i|}{N^r}$.
Thus

$$N^*_{r+1}(G) \geq \sum_{i=1}^{N}\sum_{k=1}^{N^r}\left(\left(\frac{l(h_i)(N^k_r(G)-2)}{N^r} - \frac{|c_i|}{N^r}\Big/\frac{1}{N^{r+1}}\right)-1\right) = N\alpha N^*_r(G) - N^{r+1}\mathcal{C}',$$

for all $r \in \mathbb{N}\cup\{0\}$, where

$$\mathcal{C}' = \sum_{i=1}^{N}(2\ l(h_i) + |c_i|) + 1.$$

By using the mathematical induction method, we get

$$N^*_r(G) \geq (N\alpha)^r \mathcal{C}''_m,$$

for every $m \in \mathbb{N}$ with $r > m$, where

$$\mathcal{C}''_m = \frac{1}{\alpha^m}\left(\frac{N^*_m(G)}{N^m} - \mathcal{C}'\left(\frac{1}{\alpha-1}\right)\right).$$

As, via Lemma 5.1, there exists m such that $\mathcal{C}''_m > 0$, we get

$$\underline{\dim}_B(G) \geq 1 + \log_N(\alpha). \quad \square$$

Remark 5.1. In the framework of Lemma 5.1, we have

$$1 + \frac{\log\left(\sum_{i=1}^{N} l(h_i)\right)}{\log N} \ \leq\ \underline{\dim}_B(G)\ \leq\ \overline{\dim}_B(G)\ \leq\ 1 + \frac{\log\left(\sum_{i=1}^{N} L(h_i)\right)}{\log N}.$$

Example 5.1. Let us consider the set of data points $\{(0,0),(1/4,1/3),(1/2,1/2),(3/4,1/4),(1,1)\}$ and $g_i : [0,1]\times[-1,1] \to [-1,1]$ given by

$$g_i(x,y) = c_i x + h_i(y) + d_i,$$

for every $(x,y) \in [0,1]\times[-1,1]$ and $i \in \{1,2,3,4\}$, where $h_i : [-1,1] \to [-1,1], i \in \{1,2,3,4\}$, are given by

$$h_1(y) = \frac{7}{12}y, \qquad h_2(y) = \frac{1+y}{2+y}, \qquad h_3(y) = \frac{1}{2}y^2, \qquad h_4(y) = \frac{1}{2}y,$$

for every $y \in [-1,1]$.
Therefore, we get

$$\begin{aligned}
l_1(x) &= \frac{1}{4}x, & g_1(x,y) &= \frac{-3}{12}x + \frac{7}{12}y,\\
l_2(x) &= \frac{1}{4}x + \frac{1}{4}, & g_2(x,y) &= \frac{1+y}{2+y} - \frac{1}{6},\\
l_3(x) &= \frac{1}{4}x + \frac{1}{2}, & g_3(x,y) &= \frac{-3}{4}x + \frac{1}{2}y^2 + \frac{1}{2},\\
l_4(x) &= \frac{1}{4}x + \frac{3}{4}, & g_4(x,y) &= \frac{1}{4}x + \frac{1}{2}y + \frac{1}{4},
\end{aligned}$$

for all $x \in [0,1]$ and $y \in [-1,1]$.
Note that:
- $g_i(x,y) \in [-1,1]$ for all $x \in [0,1]$ and $y \in [-1,1]$

- g_i's are Lipschitz with respect to x
- g_2 and g_3 are not Banach contractions with respect to y
- g_i's are Edelstein contractions with respect to y.

Let f^* be the corresponding contractive fractal interpolation function and G its graph.
We have:

$$\gamma = \sum_{n=1}^{4} L(h_n) = \frac{7}{12} + 1 + 1 + \frac{1}{2} = \frac{37}{12} > 1$$

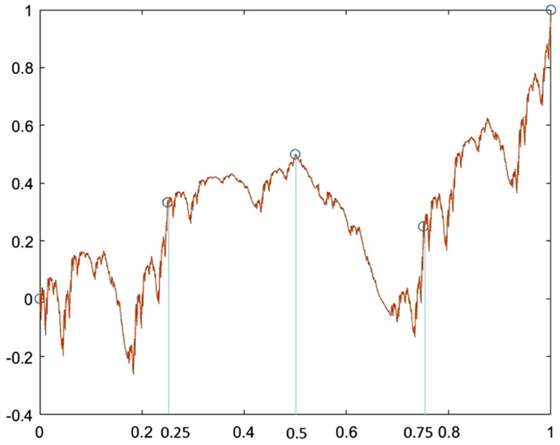
and

$$\alpha = \alpha_2 = \sum_{n=1}^{4} l(h_n) = \frac{7}{12} + \frac{1}{9} + 0 + \frac{1}{2} = \frac{43}{36} > 1,$$

since h_1, h_2, h_4 are increasing concave function and $V_1 = y_1 - (y_0 + \frac{1}{4}(y_4 - y_0)) = \frac{1}{12} > 0$.
Hence, based on Remark 5.1, we get

$$1.12817 = 1 + \log_4(43/36) \le \underline{\dim}_B(G) \le \overline{\dim}_B(G) \le \; 1 + \log_4(37/12) = 1.812245.$$

The below figure represents the graph of f^*.



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