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ORIGINAL ARTICLE

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Generalized Gibbs–Appell's equations and two-dimensional finite elements model used in flexible multibody analysis

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Abstract A planar mechanism represents a mechanism that is frequently used in engineering, and very often, the elasticity of some elements of the mechanism cannot be neglected. Consideration of all rigid elements does not allow the analysis of vibrations or situations of loss of stability of some elements. Gibbs–Appell's generalized equations are used in this paper to obtain the governing equations for a two-dimensional finite element, which is in plane motion. Using Lagrange's equations is the most widely used way for researchers to address such a problem. This is mainly due to the familiarity of researchers with this robust calculation method. There are two major advantages of applying this formalism: a smaller number of differentiation operations is needed to be performed and, by eliminating Lagrange multipliers, the number of unknowns decreases significantly. The method is applied for the plane multibody systems with elastic elements. We hope that this method, due to its simplicity, will be interesting for mechanical designers.

Keywords Elastic systems · Generalized Gibbs–Appell's equations · Finite element method · Dynamical analysis · Analytical mechanics

1 Introduction

The strong development of the industry in recent years imposes the need for complex models and projects of devices that operate at very high speeds and that withstand high forces. For this, it is necessary to develop calculation methods to provide accurate models, in the shortest possible time and at low cost. The elasticity can no longer be neglected in most engineering applications. As a result, the focus of recent research is shifting to faster, cheaper, lighter and more reliable devices for research, design and development. Applications cover areas of high interest such as light, high-speed manipulators and robots; high precision/ resolution/sensitivity spatial structures; mechanisms for high-speed applications with light elements; biodynamic systems; micro/nano-electromechanical systems.

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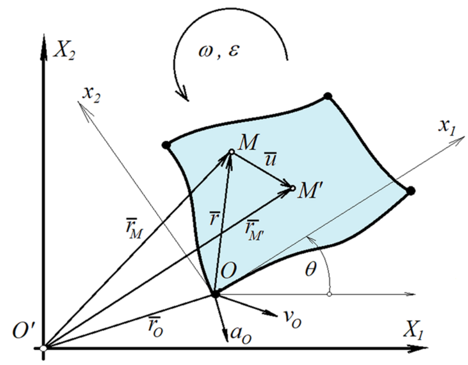

Fig. 1 Two-dimensional plane finite element

The study of multibody systems (MBS) with elastic elements involves various aspects of the study and analysis of these systems, related to the writing of equations of motion, formal analysis, solving the obtained equations, analytical study methods and numerical analysis methods. There is a lot of research in the field aimed at solving specific problems. A classification of the various research approaches can be found in the reference paper [1]. The analysis of a MBS with elastic elements follows the following steps: mathematical modeling, application of numerical methods to solve equations, use of control strategies, design of the MBS system to meet the requirements of the design theme and finally experimental validation. The first works in the field appeared quite close to the time of the development of the finite element method, in the '70s [2,3]. Industrial development has required the development of research in the field and the emergence of studies focused on certain aspects of the problem [4–11].

The most difficult part of analyzing a MBS with elastic elements is writing the equations of motion. Once these generate the use of standard procedures from the finite element method, well developed, can be used for the obtained equations. Obtaining these equations can be done in a variety of methods such as Newton-Euler's equations, Hamilton's equations, Lagrange equations, Kane and Maggi equations or other equivalent methods [12].

The Newton-Euler equations are difficult to apply in complex applications. Each element of the system must be analyzed separately, leading to numerous problems. The use of these well-known equations, although always possible, is rare in research in this field [13].

The Lagrange equation method remains the preferred method of researchers but has the disadvantage of using Lagrange multipliers, which increase the number of unknowns and, as a result, increase the number of calculations. This procedure involves a large number of operations and costly computation time [14–19].

The method of Maggi's equations has been used lately, used especially for the study of robots or manipulators. Kane's equations are another possibility offered by analytical mechanics and have become a simple choice for replacing Lagrange's equations [20–22].

First-order differential equations are obtained if Hamilton's equations are used [23–25]. All these previously presented methods as well as methods that have not been used before in analytical mechanics are currently being re-evaluated to use, in specific applications, to identify the simplest and most efficient methods, in order to obtain a low design time and costs. Following this direction is presented in this paper the energy acceleration method for plane multibody systems. It is found that this allows the advantage to obtain faster equations of motion and recently published papers use it frequently at the moment [26,27].

2 Description of the model

Consider a two-dimensional finite element. To obtain the energy of acceleration of this element, one must obtain the acceleration of a current point of an element. The point M becomes, after deformation, M' (Fig. 1).

We denote with $\bar{u}(u_1, u_2)$ the displacement of a point $M(x_1, x_2)$ of the plane. The new coordinates, after deformation, of the point are $M'(x_1', x_2')$. These coordinates are in the local frame. The following relations can be written:

$$r_{M'} = \bar{r}_M + \bar{u} = \bar{r}_O + \bar{r} + \bar{u}. \tag{1}$$

The shape functions chosen for $\bar{u}$ will determine the number of independent coordinates associated with a finite element. For reason of presenting more suggestive method, we will consider that every node will introduce two independent coordinates. The interpolation functions will be one-degree interpolation functions. The vector displacements $\bar{u}(u_1, u_2)$ can be express as:

$$u_i = N_{ij} q_j \quad i = 1, 2; \quad j = \overline{1, p}, \tag{2}$$

where N$_{ij}$ [28–30] are the shape functions and the nodal independent displacements vector $\{q\}$ is as follows:

$$\{q\}^T = \left[q_1\ q_2\ q_3\ \ldots\ \ldots\ q_p\right]. \tag{3}$$

This a very simple example but usually the plane elements are more sophisticated. The coordinates of M', expressed in the global coordinate system, are [29] as follows:

$$\begin{aligned} X_1' &= X_1 + r_{1i} u_i = X_{1o} + r_{1i} x_i + r_{1i} u_i = X_{1o} + r_{1i} x_i + r_{1i} N_{ij} \delta_j, \\ X_2' &= X_2 + r_{2i} u_i = X_{2o} + r_{2i} x_i + r_{2i} u_i = X_{2o} + r_{21} x_1 + r_{2i} N_{ij} \delta_j, i = 1, 2; \quad j = \overline{1,\ p}, \end{aligned} \tag{4}$$

or:

$$X_k' = X_{ko} + r_{ki} x_i + r_{ki} N_{ij} \delta_j, \quad k = \overline{1, 2}, \quad i = 1, 2; \quad j = \overline{1,\ p}. \tag{5}$$

Differentiating Eq. (5), it is possible to determine the velocity and acceleration:

$$\dot{X}_k' = \dot{X}_{ko} + \dot{r}_{ki} x_i + \dot{r}_{ki} N_{ij} \delta_j + r_{ki} N_{ij} \dot{\delta}_j, k = 1, 2; \quad i = 1, 2; \quad j = \overline{1,\ p}, \tag{6}$$

$$\ddot{X}_k' = \ddot{X}_{ko} + \ddot{r}_{ki} x_i + \ddot{r}_{ki} N_{ij} \delta_j + 2\dot{r}_{ki} N_{ij} \dot{\delta}_j + r_{ki} N_{ij} \ddot{\delta}_j; \quad k = 1, 2; \quad i = 1, 2; \quad j = \overline{1,\ p}. \tag{7}$$

For a single finite element, using Eq. (16), the energy of acceleration is obtained as:

$$\begin{aligned} E_{at} &= \frac{1}{2} \int_0^L \rho t \ddot{X}'_k \ddot{X}'_k \mathrm{d}A \\ &= \frac{1}{2} \int_0^L \rho t \left(\ddot{X}_{k\,o} + \ddot{r}_{ki} x_i + \ddot{r}_{ki} N_{ij} \delta_j + 2\dot{r}_{ki} N_{ij} \dot{\delta}_j + r_{ki} N_{ij} \ddot{\delta}_j\right) \\ &\quad \times \left(\ddot{X}_{ko} + \ddot{r}_{ki} x_i + \ddot{r}_{kl} N_{lm} \delta_m + 2\dot{r}_{kl} N_{lm} \dot{\delta}_m + r_{kl} N_{lm} \ddot{\delta}_m\right) \mathrm{d}A, \\ &\quad k, i, l = \overline{1, 2}; \quad j, m = \overline{1, p}. \end{aligned} \tag{8}$$

Using the notation:

$$a_{k1}(i, j) = \ddot{X}_{ko} + \ddot{r}_{ki} x_i + \ddot{r}_{ki} N_{ij} \delta_j + 2\dot{r}_{ki} N_{ij} \dot{\delta}_j, k, i = \overline{1, 2}; \quad j = \overline{1, p}, \tag{9}$$

we have:

$$E_{a1} = \frac{1}{2} \int_0^L \rho A \left(a_{k1}(i, j) + r_{ki} N_{ij} \ddot{\delta}_j\right) \left(a_{k1}(i, j) + r_{kl} N_{lm} \ddot{\delta}_m\right) \mathrm{d}A. \tag{10}$$

The coefficients r_{ij} define the position of the unit vector of the local coordinate system Oxy. The following relation can be written as follows:

$$r_{ij} r_{kj} = r_{jk} r_{ji} = \bar{\delta}_{ij}, i, j, k = 1, 2, \tag{11}$$

where $\bar{\delta}_{ij}$ is the Kronecker delta. Differentiating (11), one obtains:

$$\dot{r}_{ij} r_{kj} + r_{ij} \dot{r}_{kj} = 0, \quad i, j, k = 1, 2. \tag{12}$$

Denote:

$$\omega_{ik} = \dot{r}_{ij} r_{kj}, i, j, k = 1, 2. \tag{13}$$

Equation (12) can be expressed as follows:

$$\omega_{ik} + \omega_{ki} = 0 \quad i, j, k = 1, 2. \tag{14}$$

Equation (14) defines the angular velocity vector:

$$\omega = \omega_{21} = -\omega_{12}. \tag{15}$$

The matrix form of the operator angular velocity is as follows:

$$[\omega] = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{16}$$

The angular acceleration tensor is as follows:

$$\varepsilon_{ik} = \dot{\omega}_{ik} = \ddot{r}_{ij} r_{kj} + \dot{r}_{ij} \dot{r}_{kj} \quad i, j, k = 1, 2. \tag{17}$$

The angular acceleration is as follows:

$$\varepsilon = \varepsilon_{21} = -\varepsilon_{12}, \tag{18}$$

or in matrix form:

$$[\varepsilon] = \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{bmatrix}. \tag{19}$$

We shall have:

$$\varepsilon_{ik} = \dot{\omega}_{ik} = \ddot{r}_{ij} r_{kj} + \dot{r}_{ij} \dot{r}_{kj} = \ddot{r}_{ij} r_{kj} + \dot{r}_{ij} r_{jl} r_{ml} \dot{r}_{km} = \ddot{r}_{ij} r_{kj} - \omega_{il} \omega_{lk}, \quad i, j, k = 1, 2. \tag{20}$$

It results as follows:

$$\ddot{r}_{ij} r_{kj} = \varepsilon_{ik} + \omega_{il} \omega_{lk}, \quad i, j, k = 1, 2, \tag{21}$$

which is further used in the following.

3 Gibbs–Appell formalism

The classic Gibbs–Appell method (GA) [13] will be presented in the following. To do this, the energy of acceleration will be introduced.

A scleronomic system with p degrees of freedom, depending on p independent generalized coordinates $(q_1, q_2, \ldots, q_p)$ offers for the acceleration of an arbitrary point the expression:

$$\bar{a}_i = \sum_{k=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \bar{r}_i}{\partial q_k \partial q_j} \dot{q}_k \dot{q}_j + \sum_{k=1}^{p} \frac{\partial \bar{r}_i}{\partial q_k} \ddot{q}_k. \tag{22}$$

Introducing the acceleration (22) in the expression of the energy of acceleration, defined, for example, in [13], we obtain, generally:

$$\begin{aligned} S &= \frac{1}{2} \int_V \rho a^2 \mathrm{d}V = \frac{1}{2} \int_V \rho \left(\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j} \dot{q}_k \dot{q}_j + \sum_{k=1}^{n} \frac{\partial \bar{r}}{\partial q_k} \ddot{q}_k \right)^2 \mathrm{d}V \\ &= \frac{1}{2} \int_V \rho \left[\sum_{k=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \sum_{m=1}^{p} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j} \frac{\partial^2 \bar{r}}{\partial q_l \partial q_m} \dot{q}_k \dot{q}_j \dot{q}_l \dot{q}_m \right. \\ &\quad \left. + 2 \sum_{k=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j} \frac{\partial \bar{r}}{\partial q_l} \dot{q}_k \dot{q}_j \ddot{q}_l + \sum_{k=1}^{p} \sum_{j=1}^{p} \frac{\partial \bar{r}}{\partial q_k} \frac{\partial \bar{r}}{\partial q_j} \ddot{q}_k \ddot{q}_j \right] \mathrm{d}V \end{aligned}$$

$$= \frac{1}{2}\left(\int\limits_V \rho \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j}\frac{\partial^2 \bar{r}}{\partial q_l \partial q_m}\mathrm{d}V\right)\dot{q}_k\dot{q}_j\dot{q}_l\dot{q}_m + \left(\int\limits_V \rho \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j}\frac{\partial \bar{r}}{\partial q_l}\mathrm{d}V\right)\dot{q}_k\dot{q}_j\ddot{q}_l$$

$$+\frac{1}{2}\left(\int\limits_V \rho \sum_{k=1}^{p}\sum_{j=1}^{p} \frac{\partial \bar{r}}{\partial q_k}\frac{\partial \bar{r}}{\partial q_j}\mathrm{d}V\right)\ddot{\delta}_k\ddot{\delta}_j = E_{ao}(\dot{q}) + E_{a1}(\dot{q},\ddot{q}) + E_{a2}(\ddot{q}). \tag{23}$$

The following notations have made:

$$E_{ao}(\dot{q}) = \frac{1}{2}\left(\int\limits_V \rho \sum_{k=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p}\sum_{m=1}^{p} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j}\frac{\partial^2 \bar{r}}{\partial q_l \partial q_m}\mathrm{d}V\right)\dot{q}_k\dot{q}_j\dot{q}_l\dot{q}_m \tag{24}$$

represents the part of the energy of acceleration without accelerations,

$$E_{a1}(\dot{q},\ddot{q}) = \left(\int\limits_V \rho \sum_{k=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} \frac{\partial^2 \bar{r}}{\partial q_k \partial q_j}\frac{\partial \bar{r}}{\partial q_l}\mathrm{d}V\right)\dot{q}_k\dot{q}_j\ddot{q}_l \tag{25}$$

represents the part expressed linearly in accelerations,

$$E_{a2}(\ddot{q}) = \frac{1}{2}\left(\int\limits_V \rho \sum_{k=1}^{p}\sum_{j=1}^{p} \frac{\partial \bar{r}}{\partial q_k}\frac{\partial \bar{r}}{\partial q_j}\mathrm{d}V\right)\ddot{q}_k\ddot{q}_j, \tag{26}$$

is the part expressed quadratic in terms of accelerations.

The GA equations are obtained using the classic relations [13]:

$$\frac{\partial E_a}{\partial \ddot{q}_j} = f_j \quad j = \overline{1,p}, \tag{27}$$

where f_j is the generalized force.

In our case, the acceleration has a specific form, due to the shape function used in FEM. We adapt the Gibbs–Appell method to our case. The energy of acceleration written for a single finite element is as follows:

$$E_a = \frac{1}{2}\int_0^L \rho t\left(\ddot{X}_{ko} + \ddot{r}_{ki}\,x_i + \ddot{r}_{ki}N_{ij}\,q_j + 2\dot{r}_{ki}N_{ij}\,\dot{q}_j + r_{ki}\,N_{ij}\,\ddot{q}_j\right)$$
$$\times\left(\ddot{X}_{ko} + \ddot{r}_{kl}\,x_l + \ddot{r}_{kl}\,N_{lm}\,q_m + 2\dot{r}_{kl}\,N_{lm}\,\dot{q}_m + r_{kl}\,N_{lm}\,\ddot{q}_m\right)\mathrm{d}A. \tag{28}$$

Denoting:

$$E_{a0} = \frac{1}{2}\int_0^L \rho t\left(\ddot{X}_{ko} + \ddot{r}_{ki}x_i + \ddot{r}_{ki}N_{ij}q_j + 2\dot{r}_{ki}N_{ij}\dot{q}_j\right)\left(\ddot{X}_{mo} + \ddot{r}_{mp}x_p + \ddot{r}_{mp}N_{pr}q_r + 2\dot{r}_{mp}N_{pr}\dot{q}_r\right)\mathrm{d}A$$
$$= \frac{1}{2}\int_0^L \rho t a_k(i,j)a(l,m)\mathrm{d}A; \tag{29}$$

$$E_{a1} = \int_0^L \rho t\left(\ddot{X}_{ko} + \ddot{r}_{ki}x_i + \ddot{r}_{ki}N_{ij}q_j + 2\dot{r}_{ki}N_{ij}q_j\right)_{mp}N_{pr}\ddot{q}_r\mathrm{d}A = \int_0^L \rho t a_{k1}(i,j)N_{pr}\ddot{q}_r\mathrm{d}A; \tag{30}$$

$$E_{a2} = \frac{1}{2}\int_0^L \rho t r_{ki}r_{kl}N_{ij}N_{lm}\ddot{\delta}_j\ddot{\delta}_m\mathrm{d}A = \frac{1}{2}\int_0^L \rho t N_{ij}N_{lm}\,\ddot{\delta}_j\,\ddot{\delta}_m\,\mathrm{d}A, \tag{31}$$

it results as follows:

$$E_a = E_{a0} + E_{a1} + E_{a2}. \tag{32}$$

In Eq. (21), we have applied the relation:

$$r_{ki} r_{kl} N_{ij} N_{lm} = \bar{\delta}_{il} N_{ij} N_{lm} = N_{ij} N_{im} \quad i, l = 1, 2; \quad j, m = \overline{1, p}, \tag{33}$$

where $\bar{\delta}_{il}$ is the Kronecker delta and

$$m_{jm} = \frac{1}{2} \int_0^L \rho t N_{ij} N_{im} \mathrm{d}A, i = 1, 2; \quad j, m = \overline{1, \ p}, \tag{34}$$

are the elements of the inertial matrix. If we change the indices in (29), we obtain:

$$E_{a2} = \frac{1}{2} m_{ij} \ddot{q}_i \ddot{q}_j, \quad i, j = \overline{1, n}. \tag{35}$$

The internal energy of the element is obtained using the classic procedures [6]:

$$E_p = \frac{1}{2} q_i k_{ij} q_j, \quad i, j = \overline{1, p}. \tag{36}$$

The work done by the external concentrated loads f_i^d acting in nodes i is as follows:

$$W^c = f_j^c q_j, \quad j = \overline{1, 12}, \tag{37}$$

and the work done by the external distributed loads is as follows:

$$W^d = \int_A (p_1 u_1 + p_2 u_2) \mathrm{d}A = \left(\int_A p_k N_{ki} \mathrm{d}A \right) \delta_i = f_i^d q_i, \quad k = 1, 2; \quad i = \overline{1, p}, \tag{38}$$

where

$$f_i^d = \int_A p_k N_{ki} \mathrm{d}A \ , \quad k = 1, 2, 3, \quad i = \overline{1, p}. \tag{39}$$

The generalized external forces are as follows:

$$f_i = f_i^c + f_i^d - k_{ij} q_j, \qquad i, j = \overline{1, p}. \tag{40}$$

4 Dynamic response of a finite element

Theorem *The motion equations for two-dimensional finite element are as follows:*

$$m_{ij} \ddot{\delta}_j + 2 c_{ij}^{\omega} \dot{\delta}_j + \left(k_{ij} + k_{ij}^{\varepsilon} + k_{ij}^{\omega^2} \right) \delta_j = q_i + q_i^* - q_i^{\varepsilon} - q_i^{\omega^2} - m_{ik}^{\varepsilon} \ \varepsilon_k - m_{ik}^{o} \ddot{x}_{ko}, \tag{41}$$

where the following notations have been used:

$$m_{ij} = \frac{1}{2} \int_0^L \rho t N_{ki} N_{lj} \mathrm{d}A, \quad i, j = \overline{1, p}; \quad k, l = 1, 2; \tag{42}$$

$$c_{ij}^{\omega} = \int_0^L \rho A N_{ij} \dot{r}_{ki} r_{kl} N_{lm} \mathrm{d}x_1 = \int_0^L N_{ki} \ \omega_{km} \ N_{mj} \rho A \mathrm{d}x_1, \qquad k, i, l = 1, 2; \quad j, m = \overline{1, p}; \tag{43}$$

$$\int_0^L \rho A N_{ij} \ddot{r}_{ki} r_{kl} N_{lm} \mathrm{d}x_1 = \int_0^L \rho A N_{ij} \left(\varepsilon_{il} + \omega_{il}\omega_{lm}\right) N_{mj} \mathrm{d}x_1 = k_{ij}^{\varepsilon} + k_{ij}^{\omega}, \quad k, i, l = 1, 2; \quad j, m = \overline{1, p}; \tag{44}$$

$$k_{ij}^{\varepsilon} = \int_0^L N_{ki} \varepsilon_{km} N_{mj} \rho A \mathrm{d}x_1; \quad k_{ij}^{\omega^2} = \int_0^L N_{ki} \omega_{km} \omega_{ml} N_{lj} \rho A \mathrm{d}x_1, \quad k, m, l = 1, 2; \quad i, j = \overline{1, p}; \tag{45}$$

$$m_{ik}^{o} = \int_0^L \rho t N_{ik} \mathrm{d}A, \quad i = \overline{1, p}, \quad k = 1, 2; \tag{46}$$

$$m_{ij}^{x} = \int_0^L N_{ji} x_1 \rho t \mathrm{d}A; \quad j = \overline{1, p}, \quad i = 1, 2; \tag{47}$$

$$\int_0^L \rho A \ddot{r}_{k1} r_{kl} N_{li} x_1 \mathrm{d}x_1 = q_i^{\varepsilon} + q_i^{\omega^2}, \quad i = \overline{1, p}, \quad k, l = 1, 2. \tag{48}$$

Proof Applying the classic Gibbs–Appell equations [21], one obtains:

$$\frac{\partial E_a}{\partial \ddot{\delta}_i} - f_i = 0, \quad i = 1, 2, \tag{49}$$

from Eq. (29)), it is obtained immediately as follows:

$$\frac{\partial E_{a0}}{\partial \ddot{\delta}_i} = 0, \quad i = 1, 2. \tag{50}$$

Using Eq. (30) it results as follows:

$$\begin{aligned}
\frac{\partial E_{a1}}{\partial \ddot{\delta}_m} &= \int_0^L \rho t a_{k1} r_{kl} N_{lm} \mathrm{d}A \\
&= \int_0^L \rho t \ddot{X}_{k\,o} r_{kl} N_{lm} \mathrm{d}A + \int_0^L \rho\, t \ddot{r}_{k1} x_1 r_{kl} N_{lm} \mathrm{d}A_1 \\
&\quad + \int_0^L \rho\, t \ddot{r}_{ki} N_{ij} \delta_j r_{kl} N_{lm} \mathrm{d}A + 2 \int_0^L \rho\, t \dot{r}_{ki} N_{ij} \dot{\delta}_j r_{kl} N_{lm} \mathrm{d}A \\
&= \ddot{X}_{ko} r_{kl} \int_0^L \rho t N_{lm} \mathrm{d}A_1 + \int_0^L \rho\, t\, \ddot{r}_{k1} r_{kl} N_{lm} x_1 \mathrm{d}A_1 \\
&\quad + \int_0^L \rho\, t \ddot{r}_{ki} N_{ij} \delta_j r_{kl} N_{lm} \mathrm{d}A + 2 \left(\int_0^L \rho\, t\, N_{ij} \dot{r}_{ki} r_{kl} N_{lm} \mathrm{d}A \right) \dot{\delta}_j \\
&= m_{ik}^{o} \ddot{x}_{ko} + q_i^{\varepsilon} + q_i^{\omega^2} + \left(k_{ij}^{\varepsilon} + k_{ij}^{\omega}\right) \delta_j + 2 c_{ij}^{\omega} \dot{\delta}_j.
\end{aligned} \tag{51}$$

We have: $\ddot{x}_{ko} = r_{kj} \ddot{X}_{jo}$, $j = 1, 2;$ $k = 1, 2$, m_{ik}^{o} is defined by Eq. (40), expression $\int_0^L \rho A \ddot{r}_{k1} r_{kl} N_{lm} x_1 \mathrm{d}x_1$ by Eq. (48), k_{ij}^{ε} and $k_{ij}^{\omega^2}$ by Eq. (40), c_{ij}^{ω} by Eq. (43).

From Eqs. (31) and (42) it results as follows:

$$\frac{\partial E_{a2}}{\partial \ddot{\delta}_i} = m_{ij}\,\ddot{\delta}_j; \quad i, j = \overline{1, p}. \tag{52}$$

Using (40), (51) and (53) results the motion equations in the form (41) □

If we denote with $\left[N_{(i)}\right]$, $i = 1, 2$ the line i of the shape function matrix, and with:

$$\begin{aligned}
&[m_{11}] = \int_A N_{(1)}^T N_{(1)}\rho\, t\, \mathrm{d}A; \quad [m_{12}] = \int_A N_{(1)}^T N_{(2)}\rho\, t\, \mathrm{d}A;\\
&[m_{21}] = \int_A N_{(2)}^T N_{(1)}\rho\, t\, \mathrm{d}A; \quad [m_{22}] = \int_A N_{(2)}^T N_{(2)}\rho\, t\, \mathrm{d}A;
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\left[m_{1_1}\right] = \int_A N_{(1)}^T x_1\rho\, t\, \mathrm{d}A; \quad \left[m_{2_1}\right] = \int_A N_{(2)}^T x_1\rho\, t\, \mathrm{d}A;\\
&\left[m_{1_2}\right] = \int_A N_{(1)}^T x_2\rho\, t\, \mathrm{d}A; \quad \left[m_{2_2}\right] = \int_A N_{(2)}^T x_2\rho\, t\, \mathrm{d}A;
\end{aligned} \tag{54}$$

$$[N] = \begin{bmatrix} N_{(1)} \\ N_{(2)} \end{bmatrix}; \quad [m_O] = \int_0^L \rho t\,[N]\,\mathrm{d}A, \tag{55}$$

Equation (43) results in the following form:

$$\begin{aligned}
&([m_{11}] + [m_{22}])\{\ddot{q}\} + 2\omega([m_{12}] - [m_{21}])\{\dot{q}\} + \left[[k] + \varepsilon([m_{12}] - [m_{21}]) - \omega^2([m_{11}] + [m_{22}])\right]\{q\}\\
&\quad = \{f\} - \varepsilon\left(\left[m_{2_1}\right] - \left[m_{1_2}\right]\right) + \omega^2([m_{11}] + [m_{22}]) - [m_O][R]^T\{\ddot{r}_O\},
\end{aligned} \tag{56}$$

which is useful in engineering applications.

5 Discussion and conclusions

The standard method for obtaining equations of motion, in the case of finite element modeling of multibody systems with elastic elements, is the method of Lagrange equations. The method is widely used and familiar to the researchers. However, analytical mechanics also offers us alternative methods to the Lagrange's method, which can offer, in some cases, advantages in practical application. Thus, lately there is a frequent use of the Gibbs–Appell equation method for the analysis of such systems. The explanation of the application of the method is due to the multitude of engineering applications that require a short time of modeling and analysis. The main disadvantage of the method is that it is a method unfamiliar to researchers involving a new notion, little used, namely the energy of acceleration. But, overall, this disadvantage is offset by the fact that the number of operations required to perform is lower. The number of differentiations required for the Gibbs–Appell's method is about three times smaller than for the Lagrange method.

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