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RESEARCH ARTICLE

Diamond Intuitionistic Fuzzy Sets and Their Applications

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ABSTRACT The primary purpose of this study is to introduce diamond intuitionistic fuzzy sets ($\mathcal{D}-IFS$s). Although several generalizations of intuitionistic fuzzy sets have been introduced in the literature, the constraints of IFSs limit decision-makers ability to freely choose the degree of membership (validity, etc.) and non-membership (non-validity, etc.). The $\mathcal{D}-IFS$, however, offers this flexibility, which is why these new sets are introduced. By applying a mild restriction on the $\mathcal{D}-IFS$, a novel generalization, known as the lozenge intuitionistic fuzzy set has been acquired. Some properties of these intuitionistic fuzzy sets are characterized geometrically. The main aim of this research is also to establish some basic operations and inclusions for $\mathcal{D}-IFS$s, along with some well-known rules. On the other hand, Szmidt and Kacprzyk's forms of Hamming distance measures and, Euclidean distance measures for $\mathcal{D}-IFS$s are defined. Additionally, new score and accuracy functions are presented to compare newly defined diamond intuitionistic fuzzy relations ($\mathcal{D}-IFR$s). Moreover, two new $max-min$ operations have been derived to analyze the decomposition of $\mathcal{D}-IFR$s. Finally, a practical assessment of medical diagnosis in patients is conducted to demonstrate the feasibility and value of the proposed approaches.

INDEX TERMS Diamond intuitionistic fuzzy set, lozenge intuitionistic fuzzy set, distance measures, score function, accuracy function, $max-min$ operations.

I. INTRODUCTION

According to the norms of classical set theory, see [1], an element either belongs to a set or it does not. This creates a basic yes/no dilemma in classical logic. However, in reality, some evaluations fall between these two extremes and cannot be adequately measured using the traditional set approach. The "0–1" precision is often insufficient for identifying and addressing real-world issues. It is necessary to move beyond this binary method to better reflect how individuals perceive and interpret nature. Humans naturally make judgments that are difficult to articulate with classical logic. To address and resolve these complex issues, Zadeh [2] introduced fuzzy set theory. Fuzzy set theory facilitates the mathematical analysis of issues involving ambiguity and imprecise environments. In standard fuzzy set theory, membership degrees define the extent to which elements belong to sets, allowing a set to include elements with varying membership values. However, this approach to representing uncertainty has been deemed insufficient, see [3]. As a result, numerous extensions of fuzzy set theory have been developed over time, including Fermatean fuzzy sets, see [4], spherical fuzzy sets, see [5], hesitant fuzzy sets, see [6], Pythagorean fuzzy sets, see [7], neutrosophic fuzzy sets, see [8], and others.

The intuitionistic fuzzy set (IFS), created by Atanassov [9] in 1986, is one of the most widely used fuzzy sets. Unlike traditional fuzzy set theory, IFS incorporates notations for non-membership and indeterminacy degrees in addition to membership degrees. This feature enhances its ability to handle ambiguity and vagueness, making it valuable for problem-solving and decision-making. IFS can be integrated with various technologies and multi-criteria decision-making (MCDM) techniques, such as TOPSIS, see [10], the entropy method, see [11], MULTIMOORA, see [12], VIKOR,

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see [13], and the SIR method, see [14]. Recent and diverse industry applications found in the literature include $IF-$BWM and AHP, see [15], interval-valued intuitionistic CoCoSo, see [16], $IF-$MAIRCA, see [17], and interval-valued $IF-$AHP and WASPAS, see [18].

To expand the scope of IFS, Atanassov [19] introduced C$-IFS$ in 2020. Essentially, a C$-IFS$ represents a circle that centers on the IFS membership and non-membership degrees, with radius r. Similar to the intuitionistic fuzzy sets on which it is based, C$-IFS$ has a wide range of applications when used with MCDM techniques. Çakır et al. [20], [21] conducted research on the selection of medical waste burial sites, health tourism, and the evaluation of companies regarding industrial symbiosis. They also introduced a novel circular intuitionistic fuzzy MCDM technique. Recently, the literature has started combining C$-IFS$ with existing MCDM models. For example, in a supplier selection problem, see [22] employed C$-IFS$ with the AHP and VIKOR approaches. Similarly, Kahraman and Alkan [23] addressed uncertainty using C$-IFS$ in the application of the TOPSIS method. They conducted a supplier selection for fast-moving consumer goods (FMCG) and compared their findings with the results of the $IF-$TOPSIS method. In another study, the authors selected a waste disposal site using a methodology that integrated the VIKOR method and C$-IFS$, see [24]. Additionally, in a site selection context, the C$-IFS$ TOPSIS method was employed for selecting a site for a pandemic hospital, see [25].

We have examined IFS MCDM examples to expand the application areas of C$-IFS$, an extension of IFS. These techniques have been applied in various contexts, including personnel selection, see [26], failure mode risk assessment, see [27], solid waste disposal site selection, see [28], project criteria evaluation, see [29], healthcare waste disposal technology evaluation, [30], and bioenergy production process appraisal, [31]. Problems involving supplier evaluation and selection are well-suited for intuitionistic fuzzy MCDM techniques. For instance, Boran et al. [32] used intuitionistic fuzzy TOPSIS to help an automobile firm select the best supplier from five options. Boran [33] assessed facility placement using the same methodology, while Makui et al. [34] explored supplier selection issues with a similar approach. For a manufacturing organization, Krishankumar et al. [35] proposed an intuitionistic fuzzy PROMETHEE method for ranking alternative suppliers. Büyükozkan and Goçer [36] utilized intuitionistic fuzzy axiomatic design and intuitionistic fuzzy analytic hierarchy process (AHP) methods to evaluate sports goods brand providers in Turkey. In another study, Büyükozkan and Goçer [37] employed intuitionistic fuzzy AHP and ARAS methods to select the best supplier in a digital supply chain. Sen et al. [38] compared three intuitionistic MCDM approaches for sustainable supplier selection. For more information, related to generalizations of fuzzy sets and their applications, see [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50] and the references therein.

To explain the concept of $\mathcal{D}-IFS$, we have four objectives related to our proposed method. Due to the usage of $\mathcal{D}-IFS$s, the suggested decision-making model of the $\mathcal{D}-IFS$ is more flexible and efficient than existing techniques. In decision making problems, the $(q_1, q_2)-FS$ also classifies the input by altering the physical sense alternatives with respect to $\mathcal{D}-IF$ information. This article discusses the sources of inspiration for the proposed work in every section. This paper aims to accomplish the following several goals:

1) To define a new fuzzy set that is $\mathcal{D}-IFS$.
2) To specify a few characteristics of $\mathcal{D}-IFS$ with respect to norm "$₦_\phi$".
3) To provide a novel score functions that may be applied to diamond intuitionistic fuzzy information.
4) To define and investigate some of the properties of two newly defined operators, namely the $max-min-max$ and $max-min-min$ operators.

To create optimization $\mathcal{D}-IFS$ models in order to ascertain the attribute weights.

Motivated and inspired by ongoing research work, firstly some basic notions are recalled in Section II. In Section III, the concept of $\mathcal{D}-IFS$ has been introduced and discussed some geometric properties. Some new basic operations are also presented so that the behavior of $\mathcal{D}-IFS$ can easily understand. In Section IV, After defining $\mathcal{D}-IFR$, some new composition of these relations are also proposed. Some new score functions are introduced that is help to discuss the comparison between two $\mathcal{D}-IFR$s. Additionally, two new $max-min$ operations are defined that will help full to aggregate the data in the applications of $\mathcal{D}-IFS$. In Section V, application of $\mathcal{D}-IFS$s is presented to discuss assessment of medical diagnosis in patients and proved that these sets behavior is helpful to find to better results. Section VI present the conclusion as well as discuss about future plane.

II. PRELIMINARIES

Now we start with the basic definition of intuitionistic fuzzy set such that:

Definition 1 (Atanassov 1986): Let us have a fixed universe E and its sub-set T. The set

$$T = \{\langle \omega,\ k_T(\omega),\ s_T(\omega)\rangle \ : \text{for all } \omega \in E\}, \tag{1}$$

where $0 \le k_T(\omega) + s_T(\omega) \le 1$, is called intuitionistic fuzzy set (IFS) and functions $k_T, s_T : E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $\omega \in E$ to a fixed set $T \subseteq E$. Now, we can define also function $\pi_T : E \to [0, 1]$ by means of

$$\pi_T(\omega) = 1 - k_T(\omega) - s_T(\omega). \tag{2}$$

and it corresponds to degree of indeterminacy (uncertainty, etc.). An intuitionistic fuzzy value (IFV) is the pair "$< k_T(\omega), s_T(\omega) >$" given an element ω of X (Lim & Kim, 2005). To make things easier to understand, we can write $\tilde{\imath} =< k_{\tilde{\imath}}, s_{\tilde{\imath}} >$, where $k_{\tilde{\imath}} \in [0, 1]$, $s_{\tilde{\imath}} \in [0, 1]$ and $0 \le k_{\tilde{\imath}} + s_{\tilde{\imath}} \le 1$.

The degree of indeterminacy is represented by $\pi_{\tilde{\imath}}$, subject to the constraints that $\pi_{\tilde{\imath}} \in [0, 1]$ and $\pi_{\tilde{\imath}} = 1 - k_{\tilde{\imath}} - s_{\tilde{\imath}}$.

The definition of the complement of an *IFV* $\tilde{\imath} =< s_{\tilde{\imath}}, k_{\tilde{\imath}}, \pi_{\tilde{\imath}} >$ is as follows:

$$\tilde{\imath}^C =< s_{\tilde{\imath}}, k_{\tilde{\imath}}, \pi_{\tilde{\imath}} > . \tag{3}$$

Definition 2: Let $D[0, 1]$ denote the set of all closed subintervals of $[0, 1]$. An interval-valued intuitionistic fuzzy set $(IVIFS)\mathcal{A}$ in X is defined as $\mathcal{A} = \{< \omega, u_{\mathcal{A}}(\omega), v_{\mathcal{A}}(\omega) > \omega \in X\}$ where $u_{\mathcal{A}} : X \rightarrow D[0, 1]$ and "$v_{\mathcal{A}} : X \rightarrow D[0, 1]$ ", with the condition "$0 \leq \sup u_{\mathcal{A}}(\omega) + \sup v_{\mathcal{A}}(\omega) \leq 1, \omega \in X$". The membership and non-membership degrees of X to $\mathcal{A}$ are represented by the intervals $u_{\mathcal{A}}(\omega)$ and $v_{\mathcal{A}}(\omega)$, respectively.

An interval-valued intuitionistic fuzzy number (*IVIF*₦) is the pair $< u_{\mathcal{A}}(\omega), v_{\mathcal{A}}(\omega) >$ for any $\omega \in X$ (Atanassov & Gargov, 1989). In this study, $\tilde{\mathcal{A}} = \left(\left[u^-_{\tilde{\mathcal{A}}}, u_{\tilde{\mathcal{A}}}{}^+\right], \left[v^-_{\tilde{\mathcal{A}}}, v^+_{\tilde{\mathcal{A}}}\right]\right)$ is used to conveniently denote an *IVIF*₦. Here, $\left[u^-_{\tilde{\mathcal{A}}}, u_{\tilde{\mathcal{A}}}{}^+\right] \in D[0, 1]$, $\left[v^-_{\tilde{\mathcal{A}}}, v_{\tilde{\mathcal{A}}}{}^+\right] \in D[0, 1]$ and $u_{\tilde{\mathcal{A}}}{}^+ + v_{\tilde{\mathcal{A}}}{}^+ \leq 1$.

This new fuzzy set is an extension of the *IFS* and *IVIFS*, distinguished by a diamond representation of the degrees of membership and nonmembership.

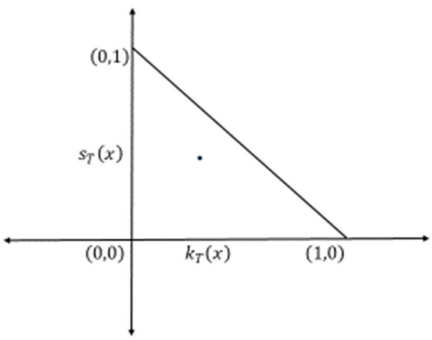

FIGURE 1. Geometric presentation of ***IFS***.

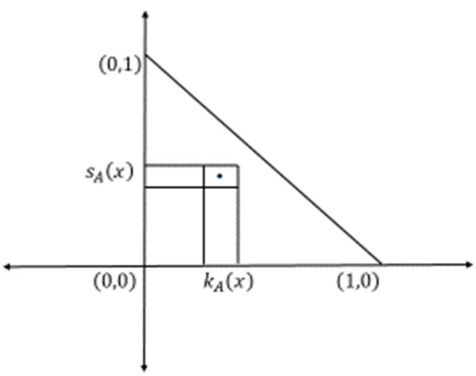

FIGURE 2. Geometric presentation of IVIFS.

III. DIAMOND INTUITIONISTIC FUZYY SETS

Now we start with the main definition of diamond intuitionistic fuzzy set such that:

Definition 3: Let us have a fixed universe E and its sub-set $\mathcal{D}$. The set

$$\mathcal{D}_{₦_\vartheta} = \{\langle \omega,\ k(\omega),\ s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}, \tag{4}$$

where $0 \leq k(\omega) + s(\omega) \leq 1$ and $₦_\vartheta \in [0,\ 2]$ is called $\mathcal{D} - IFS$ and functions $k, s : E \rightarrow [0,\ 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $\omega \in E$ to a fixed set $\mathcal{D} \subseteq E$. Now, we can define also function $\pi : E \rightarrow [0,\ 1]$ by means of

$$\pi(\omega) = 1 - k(\omega) - s(\omega).$$

and it corresponds to degree of indeterminacy (uncertainty, etc.), see Fig. 3, and Figs. 5-7. Similarly,

On the other hand $\mathcal{D}_{₦_\vartheta}$ can also be defined by using following approach such that

Let $\mathcal{D}_1 = \{\langle h, p\rangle : h, p \in [0,\ 1], \text{ and } h + p \leq 1\}$. Then,

$$\mathcal{D}_{₦_\vartheta} = \left\{\left\langle \omega,\ ₦^1(k(\omega),\ s(\omega))\right\rangle : \omega \in E\right\},$$

where

$$\begin{aligned} &₦^1(k(\omega), s(\omega)) \\ &= \left\{\langle h, p\rangle : h, p \in [0, 1] \text{ and } |k(\omega) - h| + |s(\omega) - p| \leq ₦_\vartheta\right\} \\ &\quad \cap \mathcal{D}_1, \\ &= \left\{\langle h, p\rangle : h, p \in [0, 1], |k(\omega) - h| + |s(\omega) - p| \leq ₦_\vartheta \right. \\ &\qquad \left. \times \text{ and } h + p \leq 1\right\}. \end{aligned} \tag{5}$$

The restriction of Definition 3 is presented here; however, in this instance, the intuitionistic fuzzy interpretation triangle can be entirely covered.

Definition 4: Let us have a fixed universe E and its sub-set $\mathcal{D}$. The set

$$\mathcal{D}_{₦_\vartheta} = \{\langle \omega,\ k(\omega),\ s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}, \tag{6}$$

where $0 \leq k(\omega) + s(\omega) \leq 1$ and $₦_\vartheta \in [0,\ 1]$ is called $\mathcal{D} - IFS$ and functions $k, s : E \rightarrow [0,\ 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $\omega \in E$ to a fixed set $\mathcal{D} \subseteq E$. Now, we can define also function $\pi : E \rightarrow [0,\ 1]$ by means of

$$\pi(\omega) = 1 - k(\omega) - s(\omega).$$

and it corresponds to degree of indeterminacy, see Fig. 3, Fig. 4 and Fig. 5-7.

Note that, if we want to cover the intuitionistic fuzzy interpretation triangle, then $₦_\vartheta \in [0,\ 2]$, see Fig. 4.

From Fig 3., we have

In general, the diagonals of a diamond shape are perpendicular and defined as:

$$w = 2a,$$
$$z = 2b.$$

For inradius

$$r = \frac{wz}{2\sqrt{w^2 + z^2}},$$

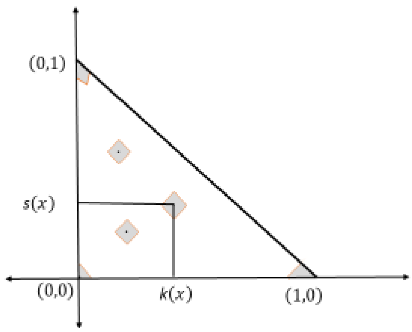

FIGURE 3. Geometrical presentation of Diamond-*IFS*.

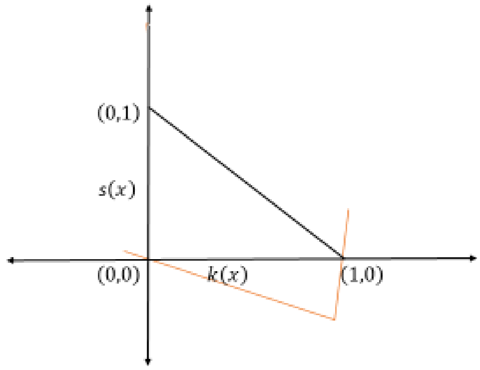

FIGURE 4. Triangular coverage of different ₦_{ϑ} values of $\mathcal{D}$-*IFS*.

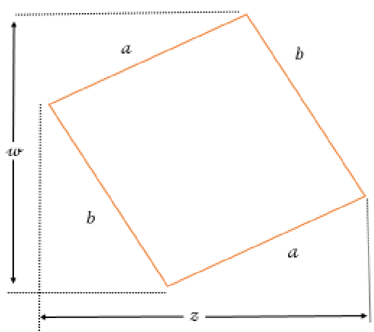

FIGURE 5. Geometrical representation of diamond with perpendicular diagonals "w" and "z".

$$= \frac{ab}{\sqrt{a^2+b^2}}.$$

From above discussion, it can be easily note that space $\mathcal{D}$-*IFS* contains the space of circular−*IFS*

Remark 1: For diamond (rhombus), the diagonals are related to the opening angle φ by, see Fig. 5.

$$w = 2a\cos\varphi,$$
$$z = 2a\sin\varphi.$$

Since diagonals of rhombus (diamond) are perpendicular and satisfy

$$w^2 + z^2 = 4a^2.$$

The rhombus is a tangential quadrilateral with $a = b = c = d$, and so has inradius, see Fig. 6:

$$r = \frac{wz}{2\sqrt{w^2+z^2}}, \\ = \frac{1}{2}a\sin(2\varphi).$$

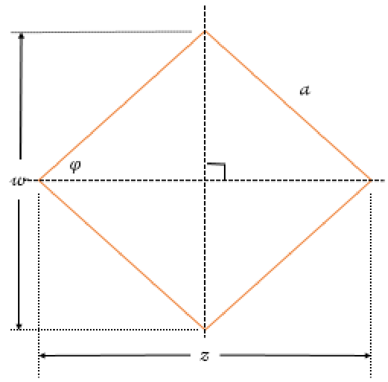

FIGURE 6. Geometrical representation of rhombus with perpendicular diagonals "w" and "z".

1) A diamond with $2\varphi = 45°$, is sometime called Lozenge. Then $\mathcal{D} - IFS$ is named as lozenge intuitionistic fuzzy set, see Fig. 7:
2) It is well known that space interval−*IFS* is greater than space *IFS*. Similarly, space $\mathcal{D}$-*IFS* contains the space of *IFS*, as well as the space of interval−*IFS*.

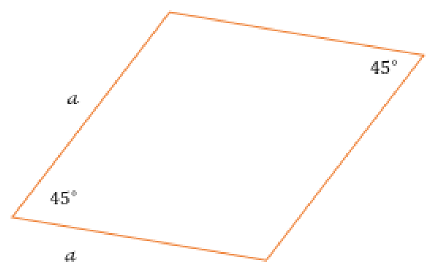

FIGURE 7. Geometrical representation of Lozenge.

A. DEVELOPMENT OF DIAMOND INTUITIONISTIC FUZZY SETS

In this section, we will discuss the procedure of calculating the $\mathcal{D} - IFS$ norm in order to convert *IFS* to $\mathcal{D} - IFS$. Using equations (7) and (8), we shall get the "₦_{ϑ}" of *IFS*.

Assume that there are intuitionistic fuzzy pairs in an *IFS* $\mathcal{D}_i$ with the following shapes: $\{(k_{i,1},\ s_{i,1}),\ (k_{i,2},\ s_{i,2}),\ (k_{i,3},\ s_{i,3}), \ldots\ldots\}$, where m is a numerical value of an *IFS* $\mathcal{D}_i$ that contains n_i, the number of intuitionistic fuzzy pairs with $\mathcal{D}_i$.

The following formula can be used to determine the "arithmetic average" of diamond fuzzy intuitionistic fuzzy pairs:

$$(k(\mathcal{D}_i), s(\mathcal{D}_i)) = \left(\sum_{j=1}^{n_i} \frac{k_{i,j}}{n_i}, \sum_{j=1}^{n_i} \frac{k_{i,j}}{n_i}\right). \quad (7)$$

The $\text{₦}_{\lozenge}$ of $(k(\mathcal{D}_i), s(\mathcal{D}_i))$ has the maximum Euclidean distance value.

$$\text{₦}_{\lozenge i} = \underset{1 \le j \le n_i}{max}\left(\left|k(\mathcal{D}_i) - k_{i,j}\right| + \left|s(\mathcal{D}_i) - s_{i,j}\right|\right), \tag{8}$$

thus, IFS is being changed into $\mathcal{D} - IFS$.

B. DISTANCE MEASURES FOR $\mathcal{D}$-IFSs

The next outcomes are introduced for representing different distances over $\mathcal{D} - IFSs$.

Definition 5: Let d be a cardinality of E. Then normalized Euclidean distance for two $\mathcal{D} - IFSs$ $\text{₤}_{\text{₦}_{\lozenge 1}}$ and $\text{₤}_{\text{₦}_{\lozenge 2}}$ is defined as

$$H_2^q\left(\text{₤}_{\text{₦}_{\lozenge 1}}, \text{₤}_{\text{₦}_{\lozenge 2}}\right) = \frac{1}{2} \times \left(\frac{\text{₦}_{\lozenge 1} - \text{₦}_{\lozenge 2}}{2} + \sqrt[q]{\frac{1}{2d}\sum\nolimits_{\omega \in E}\begin{pmatrix}|\mu_1(\omega) - \mu_2(\omega)|^q \\ + |\nu_1(\omega) - \nu_2(\omega)|^q\end{pmatrix}}\right),$$

where $q = 1, 2$. If $q = 1$ and $q = 2$, then distance $H_2\left(\text{₤}_{\text{₦}_{\lozenge 1}}, \text{₤}_{\text{₦}_{\lozenge 2}}\right)$ is known as Hamming distance and Euclidean distance for $\mathcal{D} - IFSs$, respectively.

Definition 6: Let d be a cardinality of E. Then normalized Euclidean distance for two $\mathcal{D} - IFSs$ $\text{₤}_{\text{₦}_{\lozenge 1}}$ and $\text{₤}_{\text{₦}_{\lozenge 2}}$ is defined as

$$H_3^q\left(\text{₤}_{\text{₦}_{\lozenge 1}}, \text{₤}_{\text{₦}_{\lozenge 2}}\right) = \frac{1}{2} \times \left(\frac{\text{₦}_{\lozenge 1} - \text{₦}_{\lozenge 2}}{2} + \sqrt[q]{\frac{1}{2d}\sum\nolimits_{\omega \in E}\begin{pmatrix}|\mu_1(\omega) - \mu_2(\omega)|^q \\ + |\nu_1(\omega) - \nu_2(\omega)|^q \\ + |\pi_1(\omega) - \pi_2(\omega)|^q\end{pmatrix}}\right),$$

where $q = 1, 2$. If $q = 1$ and $q = 2$, then distance $H_2\left(\text{₤}_{\text{₦}_{\lozenge 1}}, \text{₤}_{\text{₦}_{\lozenge 2}}\right)$ is known as Szmidt and Kacprzyk's form of Hamming distance and, Szmidt and Kacprzyk's form of Euclidean distance for $\mathcal{D} - IFS$s, respectively.

Note that the proof of each proposition also follows similar steps as those used in the operations of *IFSs*, and as such, is excluded for brevity.

C. BASIC OPERATIONS AND RELATIONS ON DIAMOND INTUITIONISTIC FUZZY SET

This section proposed some of the basic operations on $\mathcal{D} - IFSs$ like inclusion, union, intersection, complement, and some compositions as well as some properties are also illustrated. For the sake of easy understanding, we will take the following three $\mathcal{D} - IFSs$ over fixed universe E :

$$\text{₤}_{\text{₦}_{\lozenge 1}} = \left\{\left\langle \omega,\ k_1(\omega),\ s_1(\omega); \text{₦}_{\lozenge 1}\right\rangle\ : \text{for all } \omega \in E\right\},$$
$$\text{₤}_{\text{₦}_{\lozenge 2}} = \left\{\left\langle \omega,\ k_2(\omega),\ s_2(\omega); \text{₦}_{\lozenge 2}\right\rangle\ : \text{for all } \omega \in E\right\},$$
$$\text{₤}_{\text{₦}_{\lozenge 3}} = \left\{\left\langle \omega,\ k_3(\omega),\ s_3(\omega); \text{₦}_{\lozenge 3}\right\rangle\ : \text{for all } \omega \in E\right\}.$$

D. OPERATIONS

Here is some basic operations between two $\mathcal{D} - IFSs$ $\text{₤}_{\text{₦}_{\lozenge 1}}$ and $\text{₤}_{\text{₦}_{\lozenge 2}}$ are the following:

Definition 7: Let $\text{₤}_{\text{₦}_{\lozenge 1}}$ and $\text{₤}_{\text{₦}_{\lozenge 2}}$ be two $\mathcal{D} - IFSs$. Then,

- $\neg \text{₤}_{\text{₦}_{\lozenge 1}} = \left\{\left\langle \omega,\ s_1(\omega),\ k_1(\omega); \text{₦}_{\lozenge 1}\right\rangle\ : \text{for all } \omega \in E\right\}$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \cup_{min} \text{₤}_{\text{₦}_{\lozenge 2}} = \{\langle \omega,\ max(k_1(\omega), k_2(\omega)), min(s_1(\omega), s_2(\omega)); min\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right)\rangle\ : \text{for all } \omega \in E\}$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \cup_{max} \text{₤}_{\text{₦}_{\lozenge 2}} = \{\langle \omega,\ max(k_1(\omega), k_2(\omega)), min(s_1(\omega), s_2(\omega)); max\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right)\rangle\ : \text{for all } \omega \in E\}$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \cap_{min} \text{₤}_{\text{₦}_{\lozenge 2}} = \{\langle \omega,\ min(k_1(\omega), k_2(\omega)),\ max(s_1(\omega), s_2(\omega)); min\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right)\rangle\ : \text{for all } \omega \in E\}$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \cap_{max} \text{₤}_{\text{₦}_{\lozenge 2}} = \{\langle \omega,\ min(k_1(\omega), k_2(\omega)),\ max(s_1(\omega), s_2(\omega)); max\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right)\rangle\ : \text{for all } \omega \in E\}$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \otimes_{min} \text{₤}_{\text{₦}_{\lozenge 2}} = (k_1(\omega) \cdot k_2(\omega),\ s_1(\omega) + s_2(\omega) - s_1(\omega) \cdot s_2(\omega); min\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right))$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \otimes_{max} \text{₤}_{\text{₦}_{\lozenge 2}} = (k_1(\omega) \cdot k_2(\omega),\ s_1(\omega) + s_2(\omega) - s_1(\omega) \cdot s_2(\omega); max\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right))$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \oplus_{min} \text{₤}_{\text{₦}_{\lozenge 2}} = (k_1(\omega) + k_2(\omega) - k_1(\omega) \cdot k_2(\omega),\ s_1(\omega) \cdot s_2(\omega); min\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right))$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} \oplus_{max} \text{₤}_{\text{₦}_{\lozenge 2}} = (k_1(\omega) + k_2(\omega) - k_1(\omega) \cdot k_2(\omega),\ s_1(\omega) \cdot s_2(\omega); max\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right))$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} @_{min} \text{₤}_{\text{₦}_{\lozenge 2}} = (k_1(\omega) + k_2(\omega),\ s_1(\omega) + s_2(\omega); min\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right))$,
- $\text{₤}_{\text{₦}_{\lozenge 1}} @_{max} \text{₤}_{\text{₦}_{\lozenge 2}} = (k_1(\omega) + k_2(\omega),\ s_1(\omega) + s_2(\omega); max\left(\text{₦}_{\lozenge 1}, \text{₦}_{\lozenge 2}\right))$.

E. RELATIONS

The relations over $\mathcal{D}$-*IFSs* are firstly proposed as follows:

Definition 8: Let $\text{₤}_{\text{₦}_{\lozenge 1}}$ and $\text{₤}_{\text{₦}_{\lozenge 2}}$ be two $\mathcal{D} - IFSs$. Then, for all $\omega \in E$, we have

- $\text{₤}_{\text{₦}_{\lozenge 1}} \subset_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff
- $\left(\begin{pmatrix}\text{₦}_{\lozenge 1} \\ = \text{₦}_{\lozenge 2}\end{pmatrix} \& \begin{pmatrix}(k_1(\omega) < k_2(\omega)\ \&\ s_1(\omega) \ge s_2(\omega)) \\ \vee (k_1(\omega) \le k_2(\omega)\ \&\ s_1(\omega) > s_2(\omega)) \\ \vee (k_1(\omega) < k_2(\omega)\ \&\ s_1(\omega) > s_2(\omega))\end{pmatrix}\right)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subset_{\rho} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\left(\left(\text{₦}_{\lozenge 1} < \text{₦}_{\lozenge 2}\right) \& k_1(\omega) = k_2(\omega)\ \&\ s_1(\omega) = s_2(\omega)\right)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subset \text{₤}_{\text{₦}_{\lozenge 2}}$ iff
- $\left(\begin{pmatrix}\text{₦}_{\lozenge 1} \\ < \text{₦}_{\lozenge 2}\end{pmatrix} \& \begin{pmatrix}(k_1(\omega) < k_2(\omega)\ \&\ s_1(\omega) \ge s_2(\omega)) \\ \vee (k_1(\omega) \le k_2(\omega)\ \&\ s_1(\omega) > s_2(\omega)) \\ \vee (k_1(\omega) < k_2(\omega)\ \&\ s_1(\omega) > s_2(\omega))\end{pmatrix}\right)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subset_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₤}_{\text{₦}_{\lozenge 2}} \supset_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 1}}$
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subset_{\rho} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₤}_{\text{₦}_{\lozenge 2}} \supset_{\rho} \text{₤}_{\text{₦}_{\lozenge 1}}$
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subset \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₤}_{\text{₦}_{\lozenge 2}} \supset \text{₤}_{\text{₦}_{\lozenge 1}}$
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subseteq_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\left(\left(\text{₦}_{\lozenge 1} = \text{₦}_{\lozenge 2}\right) \& k_1(\omega) \le k_2(\omega)\ \&\ s_1(\omega) \ge s_2(\omega)\right)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subseteq_{\rho} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\left(\left(\text{₦}_{\lozenge 1} \le \text{₦}_{\lozenge 2}\right) \& k_1(\omega) = k_2(\omega)\ \&\ s_1(\omega) = s_2(\omega)\right)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subseteq \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\left(\left(\text{₦}_{\lozenge 1} \le \text{₦}_{\lozenge 2}\right) \& k_1(\omega) \le k_2(\omega)\ \&\ s_1(\omega) \ge s_2(\omega)\right)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subseteq_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₤}_{\text{₦}_{\lozenge 2}} \supseteq_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 1}}$
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subseteq_{\rho} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₤}_{\text{₦}_{\lozenge 2}} \supseteq_{\rho} \text{₤}_{\text{₦}_{\lozenge 1}}$
- $\text{₤}_{\text{₦}_{\lozenge 1}} \subseteq \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₤}_{\text{₦}_{\lozenge 2}} \supseteq \text{₤}_{\text{₦}_{\lozenge 1}}$
- $\text{₤}_{\text{₦}_{\lozenge 1}} =_{\upsilon} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $k_1(\omega) = k_2(\omega)\ \&\ s_1(\omega) = s_2(\omega)$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} =_{\rho} \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\text{₦}_{\lozenge 1} = \text{₦}_{\lozenge 2}$;
- $\text{₤}_{\text{₦}_{\lozenge 1}} = \text{₤}_{\text{₦}_{\lozenge 2}}$ iff $\left(\text{₦}_{\lozenge 1} = \text{₦}_{\lozenge 2}\right) \& k_1(\omega) = k_2(\omega)\ \&\ s_1(\omega) = s_2(\omega)$.

From Definition 7 and Definition 8, we have concluded the following results:

Proposition 1: Let $\pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_2}}$ be two $\mathcal{D}-IFSs$. Then, following properties holds such that

- $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_2}}$, and $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}max \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_2}}$,
- $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_2}}$, and $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_2}}$,
- $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_2}}$, and $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_2}}$,
- $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_2}}$, and $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_2}}$,
- $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_2}}$, and $\neg\left(\neg\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \neg\pounds_{\mathbb{N}_{\theta_2}}\right) = \pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_2}}$.

Proposition 2: Let $\pounds_{\mathbb{N}_{\theta_1}}$, $\pounds_{\mathbb{N}_{\theta_2}}$ and $\pounds_{\mathbb{N}_{\theta_3}}$ be three $\mathcal{D}-IFSs$. Then, following properties holds such that

1) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_1}} = \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_1}} = \pounds_{\mathbb{N}_{\theta_1}}$,
2) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_1}} = \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_1}} = \pounds_{\mathbb{N}_{\theta_1}}$,
3) $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_1}} = \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_1}} = \pounds_{\mathbb{N}_{\theta_1}}$,
4) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_1}}$,
5) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_1}}$,
6) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_1}}$,
7) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_1}}$,
8) $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{@}min \pounds_{\mathbb{N}_{\theta_1}}$, and $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_2}} = \pounds_{\mathbb{N}_{\theta_2}} {}^{@}max \pounds_{\mathbb{N}_{\theta_1}}$,
9) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}$,
10) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cap}min \pounds_{\mathbb{N}_{\theta_3}}$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cap}max \pounds_{\mathbb{N}_{\theta_3}}$,
11) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_3}}$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_3}}$,
12) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_3}}$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_3}}$,
13) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
14) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\cap}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
15) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
16) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\oplus}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
17) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
18) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{\otimes}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
19) $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}min \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}min \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
20) $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}min \pounds_{\mathbb{N}_{\theta_3}}\right)$ and $\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cup}max \pounds_{\mathbb{N}_{\theta_3}}\right) = \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_2}}\right) {}^{\cup}max \left(\pounds_{\mathbb{N}_{\theta_1}} {}^{@}max \pounds_{\mathbb{N}_{\theta_3}}\right)$,
21) $\pounds_{\mathbb{N}_{\theta_1}} {}^{\cup}min\left(\pounds_{\mathbb{N}_{\theta_2}} {}^{\cap}min \pounds_{\mathbb{N}_{\theta_3}}\right)$

$$= \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

22) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\cup}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

23) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

24) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

25) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

26) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

27) $$£_{\mathbf{N}_{\vartheta_1}} {}^{@}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{@}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

28) $$£_{\mathbf{N}_{\vartheta_1}} {}^{@}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{@}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\cap}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\cap}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

Proposition 3: Let $£_{\mathbf{N}_{\vartheta_1}}, £_{\mathbf{N}_{\vartheta_2}}$ and $£_{\mathbf{N}_{\vartheta_3}}$ be three $\mathcal{D}-IFSs$. Then, following properties holds such that

a) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

b) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

c) $$£_{\mathbf{N}_{\vartheta_1}} {}^{@}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{@}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

d) $$£_{\mathbf{N}_{\vartheta_1}} {}^{@}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{@}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right) \subset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

e) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

f) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

g) $$£_{\mathbf{N}_{\vartheta_1}} {}^{@}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{@}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}min £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

h) $$£_{\mathbf{N}_{\vartheta_1}} {}^{@}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{@}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{\otimes}max £_{\mathbf{N}_{\vartheta_3}}\right) \supset \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{\otimes}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

i) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{@}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{@}min \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

j) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_2}}\right) {}^{@}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}min £_{\mathbf{N}_{\vartheta_3}}\right) \text{ and}$$
$$£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max \left(£_{\mathbf{N}_{\vartheta_2}} {}^{@}max £_{\mathbf{N}_{\vartheta_3}}\right) = \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_2}}\right) {}^{@}max \left(£_{\mathbf{N}_{\vartheta_1}} {}^{\oplus}max £_{\mathbf{N}_{\vartheta_3}}\right).$$

k) $$£_{\mathbf{N}_{\vartheta_1}} {}^{\otimes}min \left(£_{\mathbf{N}_{\vartheta_2}} {}^{@}min £_{\mathbf{N}_{\vartheta_3}}\right)$$

$= \left(\pounds_{₦_{\vartheta_1}} \otimes_{min} \pounds_{₦_{\vartheta_2}}\right) @_{min} \left(\pounds_{₦_{\vartheta_1}} \otimes_{min} \pounds_{₦_{\vartheta_3}}\right)$ and

$\pounds_{₦_{\vartheta_1}} \otimes_{max} \left(\pounds_{₦_{\vartheta_2}} @_{min} \pounds_{₦_{\vartheta_3}}\right)$

$= \left(\pounds_{₦_{\vartheta_1}} \otimes_{max} \pounds_{₦_{\vartheta_2}}\right) @_{min} \left(\pounds_{₦_{\vartheta_1}} \otimes_{max} \pounds_{₦_{\vartheta_3}}\right)$.

l) $\pounds_{₦_{\vartheta_1}} \otimes_{min} \left(\pounds_{₦_{\vartheta_2}} @_{max} \pounds_{₦_{\vartheta_3}}\right)$

$= \left(\pounds_{₦_{\vartheta_1}} \otimes_{min} \pounds_{₦_{\vartheta_2}}\right) @_{max} \left(\pounds_{₦_{\vartheta_1}} \otimes_{min} \pounds_{₦_{\vartheta_3}}\right)$ and

$\pounds_{₦_{\vartheta_1}} \otimes_{max} \left(\pounds_{₦_{\vartheta_2}} @_{max} \pounds_{₦_{\vartheta_3}}\right)$

$= \left(\pounds_{₦_{\vartheta_1}} \otimes_{max} \pounds_{₦_{\vartheta_2}}\right) @_{max} \left(\pounds_{₦_{\vartheta_1}} \otimes_{max} \pounds_{₦_{\vartheta_3}}\right)$.

m) $\pounds_{₦_{\vartheta_1}} \otimes_{min} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} \cap_{min} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} @_{min} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} \cup_{min} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} \oplus_{min} \pounds_{₦_{\vartheta_2}}$,

n) $\pounds_{₦_{\vartheta_1}} \otimes_{max} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} \cap_{max} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} @_{max} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} \cup_{max} \pounds_{₦_{\vartheta_2}} \subseteq_{\upsilon} \pounds_{₦_{\vartheta_1}} \oplus_{max} \pounds_{₦_{\vartheta_2}}$.

Note that these new outcomes Proposition 1, Proposition 2, Proposition 3, and Preposition 4 can be easily proved as similar manner for the sets.

Remark 2: If we take $₦_{\vartheta_1} = 0 = ₦_{\vartheta_2} = ₦_{\vartheta_3}$, the all operations and relations reduce for $\mathcal{D}-IFSs$ as well as $IFSs$.

F. DIAMOND INTUITIONISTIC FUZZY MODEL OPERATORS

In this sections, some of the new model operators are introduced using the approach of intuitionistic fuzzy approach as well as similar to logic operators "necessity" and "possibility". Moreover, some extensions are also obtained with the help of some parameters. Firstly, we start with these two operators such that:

Definition 9: Let $\pounds_{₦_\vartheta}$ be a $\mathcal{D}-IFS$. Then, we have

$$\Box \pounds_{₦_\vartheta} = \{\langle \omega, k(\omega), 1-k(\omega); ₦_\vartheta \rangle \mid \omega \in E\} = \{\langle \omega, ₦^1(k(\omega), 1-k(\omega)) \rangle \mid \omega \in E\}. \quad (9)$$

$$\Diamond \pounds_{₦_\vartheta} = \{\langle \omega, 1-s(\omega), s(\omega); ₦_\vartheta \rangle \mid \omega \in E\} = \{\langle \omega, ₦^1(1-s(\omega), s(\omega)) \rangle \mid \omega \in E\}. \quad (10)$$

Let $\omega, \gamma \in [0, 1]$ be fixed numbers. Then, following are the extensions of diamond intuitionistic fuzzy model operators:

$D_\omega(\pounds_{₦_\vartheta})$
$= \{\langle \omega, k(\omega) + \omega\pi(\omega), s(\omega) + (1-\omega)\pi(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(k(\omega) + \omega\pi(\omega), s(\omega) + (1-\omega)\pi(\omega)) \rangle \mid \omega \in E\}$,

$F_{\omega,\gamma}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, k(\omega) + \omega\pi(\omega), s(\omega) + \beta\pi(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(k(\omega) + \omega\pi(\omega), s(\omega) + \beta\pi(\omega)) \rangle \mid \omega \in E\}$,

$G_{\omega,\gamma}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, \omega k(\omega), \beta s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(\omega k(\omega), \beta s(\omega)) \rangle \mid \omega \in E\}$,

$H_{\omega,\gamma}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, \omega k(\omega), s(\omega) + \beta\pi(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(\omega k(\omega), s(\omega) + \beta\pi(\omega)) \rangle \mid \omega \in E\}$,

$H^*_{\omega,\gamma}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, \omega k(\omega), s(\omega) + \beta(1 - \omega k(\omega) - s(\omega)); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(\omega k(\omega), s(\omega) + \beta(1 - \omega k(\omega) - s(\omega))) \rangle \mid \omega \in E\}$,

$\overline{H_{\omega,\gamma}}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, \omega k(\omega), s(\omega) + \beta - \beta s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(\omega k(\omega), s(\omega) + \beta - \beta s(\omega)) \rangle \mid \omega \in E\}$,

$J_{\omega,\gamma}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, k(\omega) + \omega\pi(\omega), \beta s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(k(\omega) + \omega\pi(\omega), \beta s(\omega)) \rangle \mid \omega \in E\}$,

$J^*_{\omega,\gamma}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, \frac{1}{2}(k(\omega) + \omega(1 - k(\omega) - \beta s(\omega))), \beta s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1\left(\frac{1}{2}(k(\omega) + \omega(1 - k(\omega) - \beta s(\omega))), \beta s(\omega)\right) \rangle \mid \omega \in E\}$,

$\overline{J_{\omega,\gamma}}(\pounds_{₦_\vartheta})$
$= \{\langle \omega, k(\omega) + \omega - \omega k(\omega), \beta s(\omega); ₦_\vartheta \rangle \mid \omega \in E\}$
$= \{\langle \omega, ₦^1(k(\omega) + \omega - \omega k(\omega), \beta s(\omega)) \rangle \mid \omega \in E\}$.

IV. RELATION ON DIAMOND INTUITIONISTIC FUZZY RELATIONS

The relations over different fuzzy sets have be introduced because these relations are widely applied in real-world tasks like computations and diagnostic tests for medical conditions. $\mathcal{D}-IFR$s are the topic of this section. Additionally, covered are the characteristics and compositions of established relations, which are subsequently used to apply a diamond intuitionistic fuzzy framework to a medical diagnostics problem.

Definition 10: Let us have a fixed universe $X \times Y$ and its sub-set $₣_\vartheta$. The set

$$₣_\vartheta = \left\{(\omega, \Lambda), k_{₣_\vartheta}(\omega, \Lambda), s_{₣_\vartheta}(\omega, \Lambda); ₦_\vartheta \,\middle|\, \omega \in X \text{ and } \Lambda \in Y\right\}, \quad (11)$$

where $0 \leq k_{₣_\vartheta}(\omega, \Lambda) + s_{₣_\vartheta}(\omega, \Lambda) \leq 1$ and $₦_\vartheta \in [0, 2]$, is called $\mathcal{D}-IFR$ and functions $k_{₣_\vartheta}, s_{₣_\vartheta} : X \times Y \to [0, 1]$.

Now, we can define also function $\pi : E \to [0, 1]$ by means of

$$\pi_{₣_\vartheta}(\omega, \Lambda) = 1 - \left(k_{₣_\vartheta}(\omega, \Lambda) + s_{₣_\vartheta}(\omega, \Lambda)\right). \quad (12)$$

and it corresponds to degree of indeterminacy (uncertainty, etc.).Furthermore, $\mathcal{D}-IFR(X \times Y)$ represents the set of all $\mathcal{D}$-$IFRs$ on $X \times Y$.

Definition 11: For any relation $₣_\vartheta$ over $X \times Y$, we present inverse relation of $₣_\vartheta$, $₣_\vartheta^{-1}$ over $X \times Y$ as $k_{₣_\vartheta}(\omega, \Lambda) = k_{₣_\vartheta^{-1}}(\Lambda, \omega)$, $s_{₣_\vartheta}(\omega, \Lambda) = s_{₣_\vartheta^{-1}}(\Lambda, \omega)$ and $₦_{₣_{₦_\vartheta}} = ₦_{₣^{-1}_{₦_\vartheta}} = ₦_\vartheta$.

Now we characterize some properties of $\mathcal{D}-IFR$ such that

Definition 12: Let $₣_{\vartheta_1}$ and $₣_{\vartheta_2}$ be two $\mathcal{D}-IFRs$. Then,

∘ $\neg ₣_{\vartheta_1}$

$= \left\{ (\omega, \Lambda), s_{₣_{\vartheta_1}}(\omega, \Lambda), k_{₣_{\vartheta_1}}(\omega, \Lambda); ₦_{\vartheta_1} \text{ : for all } (\omega, \Lambda) \in X \times Y \right\}$,

- $₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} = \left\{ (\omega, \Lambda), \left(k_{₣_{\vartheta_1}}(\omega, \Lambda) \vee k_{₣_{\vartheta_2}}(\omega, \Lambda) \right), \left(s_{₣_{\vartheta_1}}(\omega, \Lambda) \wedge s_{₣_{\vartheta_2}}(\omega, \Lambda) \right); \left(₦_{\vartheta_1} \wedge ₦_{\vartheta_2} \right) \text{ : for all } (\omega, \Lambda) \in X \times Y \right\}$,
- $₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2} = \left\{ (\omega, \Lambda), \left(k_{₣_{\vartheta_1}}(\omega) \vee k_{₣_{\vartheta_2}}(\omega) \right), \left(s_{₣_{\vartheta_1}}(\omega) \wedge s_{₣_{\vartheta_2}}(\omega) \right); \left(₦_{\vartheta_1} \vee ₦_{\vartheta_2} \right) \text{ : for all } (\omega, \Lambda) \in X \times Y \right\}$,
- $₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} = \left\{ \left\langle (\omega, \Lambda), \ min\left(k_{₣_{\vartheta_1}}(\omega) \wedge k_{₣_{\vartheta_2}}(\omega) \right), \ max \left(s_{₣_{\vartheta_1}}(\omega) \vee s_{₣_{\vartheta_2}}(\omega) \right); min\left(₦_{\vartheta_1} \wedge ₦_{\vartheta_2} \right) \right\rangle \text{ : for all } (\omega, \Lambda) \in X \times Y \right\}$,
- $₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} = \left\{ \left\langle (\omega, \Lambda), \left(k_{₣_{\vartheta_1}}(\omega, \Lambda) \wedge k_{₣_{\vartheta_2}}(\omega, \Lambda) \right), \left(s_{₣_{\vartheta_1}}(\omega, \Lambda) \vee s_{₣_{\vartheta_2}}(\omega, \Lambda) \right); \left(₦_{\vartheta_1}, ₦_{\vartheta_2} \right) \right\rangle \text{ : for all } (\omega, \Lambda) \in X \times Y \right\}$,
- $₣_{\vartheta_1} \leq_\upsilon ₣_{\vartheta_2}$ iff $\left(\left(₦_{\vartheta_1} = ₦_{\vartheta_2} \right) \ \& \ k_{₣_{\vartheta_1}}(\omega) \leq k_{₣_{\vartheta_2}}(\omega) \ \& \ s_{₣_{\vartheta_1}}(\omega) \geq s_{₣_{\vartheta_2}}(\omega) \right)$,
- $₣_{\vartheta_1} \leq_\rho ₣_{\vartheta_2}$ iff $\left(\left(₦_{\vartheta_1} \leq ₦_{\vartheta_2} \right) \ \& \ k_{₣_{\vartheta_1}}(\omega) = k_{₣_{\vartheta_2}}(\omega) \ \& \ s_{₣_{\vartheta_1}}(\omega) = s_{₣_{\vartheta_2}}(\omega) \right)$,
- $₣_{₣_{\vartheta_1}} \leq ₣_{₣_{\vartheta_2}}$ iff $\left(\left(₦_{\vartheta_1} \leq ₦_{\vartheta_2} \right) \ \& \ k_{₣_{\vartheta_1}}(\omega) \leq k_{₣_{\vartheta_2}}(\omega) \ \& \ s_{₣_{\vartheta_1}}(\omega) \geq s_{₣_{\vartheta_2}}(\omega) \right)$.

Proposition 4: Let $₣_{\vartheta_1}$, $₣_{\vartheta_2}$ and $₣_{\vartheta_1}$ be three $\mathcal{D} - IFRs$. Then, following properties holds such that

1) $₣_{\vartheta_1} \leq ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_1}^{-1} \leq ₣_{\vartheta_2}^{-1}$,
2) $₣_{\vartheta_1} \leq_\upsilon ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_1}^{-1} \leq ₣_{\vartheta_2}^{-1}$,
3) $₣_{\vartheta_1} \leq_\rho ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_1}^{-1} \leq ₣_{\vartheta_2}^{-1}$,
4) $\left(₣_{\vartheta_1}^{-1} \right)^{-1} = ₣_{\vartheta_1}$,
5) $\left(₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \right)^{-1} = ₣_{\vartheta_1}^{-1} \wedge_{min} ₣_{\vartheta_2}^{-1}$
6) $\left(₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \right)^{-1} = ₣_{\vartheta_1}^{-1} \wedge_{max} ₣_{\vartheta_2}^{-1}$
7) $₣_{\vartheta_1} \wedge_{min} \left(₣_{\vartheta_2} \vee_{min} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \right) \vee_{min} \left(₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_3} \right)$ and $₣_{\vartheta_1} \wedge_{max} \left(₣_{\vartheta_2} \vee_{min} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \right) \vee_{min} \left(₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_3} \right)$,
8) $₣_{\vartheta_1} \wedge_{min} \left(₣_{\vartheta_2} \vee_{max} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \right) \vee_{max} \left(₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_3} \right)$ and $₣_{\vartheta_1} \wedge_{max} \left(₣_{\vartheta_2} \vee_{max} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \right) \vee_{max} \left(₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_3} \right)$,
9) $₣_{\vartheta_1} \vee_{min} \left(₣_{\vartheta_2} \wedge_{min} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} \right) \wedge_{min} \left(₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_3} \right)$ and $₣_{\vartheta_1} \vee_{max} \left(₣_{\vartheta_2} \wedge_{min} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2} \right) \wedge_{min} \left(₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_3} \right)$,
10) $₣_{\vartheta_1} \vee_{min} \left(₣_{\vartheta_2} \wedge_{max} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} \right) \wedge_{max} \left(₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_3} \right)$ and $₣_{\vartheta_1} \vee_{max} \left(₣_{\vartheta_2} \wedge_{max} ₣_{\vartheta_3} \right) = \left(₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2} \right) \wedge_{max} \left(₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_3} \right)$,
11) $₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \leq ₣_{\vartheta_1}$
12) $₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \leq ₣_{\vartheta_1}$
13) $₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \leq ₣_{\vartheta_2}$
14) $₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \leq ₣_{\vartheta_2}$
15) $₣_{\vartheta_1} \wedge_{min} ₣_{₦_2} \leq ₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2}$,
16) $₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} \leq ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2}$.
17) $₣_{\vartheta_1} \leq_\upsilon ₣_{\vartheta_3} \& ₣_{\vartheta_2} \leq_\upsilon ₣_{\vartheta_3} \Rightarrow ₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \leq_\upsilon ₣_{\vartheta_3}$ and $₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \leq_\upsilon ₣_{\vartheta_3}$.
18) $₣_{\vartheta_1} \leq_\upsilon ₣_{\vartheta_3} \& ₣_{\vartheta_2} \leq_\upsilon ₣_{\vartheta_3} \Rightarrow ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} \leq_\upsilon ₣_{\vartheta_3}$ and $₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2} \leq_\upsilon ₣_{\vartheta_3}$.
19) $₣_{\vartheta_1} \leq_\rho ₣_{\vartheta_3} \& ₣_{\vartheta_2} \leq_\rho ₣_{\vartheta_3} \Rightarrow ₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \leq_\rho ₣_{\vartheta_3}$ and $₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \leq_\rho ₣_{\vartheta_3}$.
20) $₣_{\vartheta_1} \leq_\rho ₣_{\vartheta_3} \& ₣_{\vartheta_2} \leq_\rho ₣_{\vartheta_3} \Rightarrow ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} \leq_\rho ₣_{\vartheta_3}$ and $₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2} \leq_\rho ₣_{\vartheta_3}$.
21) $₣_{\vartheta_1} \leq ₣_{\vartheta_3} \& ₣_{\vartheta_2} \leq ₣_{\vartheta_3} \Rightarrow ₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2} \leq ₣_{\vartheta_3}$ and $₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2} \leq ₣_{\vartheta_3}$.
22) $₣_{\vartheta_1} \leq ₣_{\vartheta_3} \& ₣_{\vartheta_2} \leq ₣_{\vartheta_3} \Rightarrow ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2} \leq ₣_{\vartheta_3}$ and $₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2} \leq ₣_{\vartheta_3}$.
23) $₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_1} \& ₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2}$ and $₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2}$.
24) $₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_1} \& ₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2}$ and $₣_{\vartheta_3} \leq_\upsilon ₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2}$.
25) $₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_1} \& ₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2}$ and $₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2}$.
26) $₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_1} \& ₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2}$ and $₣_{\vartheta_3} \leq_\rho ₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2}$.
27) $₣_{\vartheta_3} \leq ₣_{\vartheta_1} \& ₣_{\vartheta_3} \leq ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_3} \leq ₣_{\vartheta_1} \wedge_{min} ₣_{\vartheta_2}$ and $₣_{\vartheta_3} \leq ₣_{\vartheta_1} \wedge_{max} ₣_{\vartheta_2}$.
28) $₣_{\vartheta_3} \leq ₣_{\vartheta_1} \& ₣_{\vartheta_3} \leq ₣_{\vartheta_2} \Rightarrow ₣_{\vartheta_3} \leq ₣_{\vartheta_1} \vee_{min} ₣_{\vartheta_2}$ and $₣_{\vartheta_3} \leq ₣_{\vartheta_1} \vee_{max} ₣_{\vartheta_2}$.

Proof.

1). Let $₣_{\vartheta_1} \leq ₣_{\vartheta_2}$ to prove $₣_{\vartheta_1}^{-1} \leq ₣_{\vartheta_2}^{-1}$. Since $₣_{\vartheta_1} \leq ₣_{\vartheta_2}$, then by Definition 11, we have $k_{₣_{\vartheta_1}^{-1}}(\Lambda, \omega) = k_{₣_{\vartheta_1}}(\omega, \Lambda) \leq k_{₣_{\vartheta_2}}(\omega, \Lambda) = k_{₣_{\vartheta_2}^{-1}}(\Lambda, \omega)$,. Also $s_{₣_{\vartheta_1}^{-1}}(\Lambda, \omega) = s_{₣_{\vartheta_1}}(\omega, \Lambda) \leq s_{₣_{\vartheta_2}}(\omega, \Lambda) = s_{₣_{\vartheta_2}^{-1}}(\Lambda, \omega)$ and $₦_{\vartheta_1} \leq ₦_{\vartheta_2}$. Hence, $₣_{\vartheta_1}^{-1} \leq ₣_{\vartheta_2}^{-1}$. 4). As $k_{\left(₣_{\vartheta_1}^{-1}\right)^{-1}}(\omega, \Lambda) = k_{₣_{\vartheta_1}^{-1}}(\Lambda, \omega) = k_{₣_{\vartheta_1}}(\omega, \Lambda)$, $s_{\left(₣_{\vartheta_1}^{-1}\right)^{-1}}(\omega, \Lambda) = s_{₣_{\vartheta_1}^{-1}}(\Lambda, \omega) = s_{₣_{\vartheta_1}}(\omega, \Lambda)$ and $₦_{₣_{\vartheta_1}} = ₦_{₣_{\vartheta_1}^{-1}} = ₦_{\vartheta_1}$. Hence, $\left(₣_{\vartheta_1}^{-1}\right)^{-1} = ₣_{\vartheta_1}$.

Analogously, other outcomes can be proved.

Next we will discuss the decomposition of $\mathcal{D} - IFRs$ in the sense of $max - min$ operations.

Definition 13: Let $₣_{\vartheta_1} \in \mathcal{D}$-*IFR* $(X \times Y)$ and $₣_{\vartheta_2} \in \mathcal{D}$-*IFR* $(Y \times Z)$ be two $\mathcal{D}$-*IFRs*. Then, $max-min-max$ and $max-min-min$ composition of two relations is denoted by $₣_{\vartheta_1} C_{max} ₣_{\vartheta_2}$ and $₣_{\vartheta_1} C_{min} ₣_{\vartheta_2}$, respectively, and defined as (13), shown at the bottom of the next page.

Definition 14: The degree of affiliation between components $(\omega, \text{П})$ of $\mathcal{D}$-*IFR* $₣_\vartheta : X \rightarrow Z$ is defined as follows with respect to the decision-maker's preference information $\lambda \in [0, 1]$ (14), as shown at the bottom of the next page, where $\pi_{₣_\vartheta}(\omega, q) = 1 - \left(k_{₣_\vartheta}(\omega, q) + s_{₣_\vartheta}(\omega, q) \right)$.

Note that when $₦_\vartheta = 0$, then $SC^*_{₣}(\omega, q)$ reduces to score function for *IFR*.

V. APPLICATIONS

This section discusses several real-world issues with diamond intuitionistic fuzzy environments, such as making decisions and diagnosing medical conditions. The proposed $\mathcal{D}-IFSs$ concepts are sufficiently versatile that they can be used to solve a wide range of uncertainty situations. $\mathcal{D}-IFSs$ are currently used in medical diagnosis and decision-making scenarios.

A. APPLICATIONS OF $\mathcal{D}$-IFSs IN EVALUATIONS FOR MEDICAL DIAGNOSIS

Regarding their practical applications of $\mathcal{D}-IFSs$, the theory on relations of fuzzy set theory is crucial. This section discusses the use of $\mathcal{D}-IFSs$ in medical diagnostics. The $\mathcal{D}-IFR$ ₣ $: P \rightarrow D$, which characterizes a patient's affiliation with a diagnosis, is defined as "diamond medical knowledge". Here, P stands for the set of patients, and D for the multiple diagnoses. Furthermore, Q represents the collection of symptoms experienced by the patients. Fig. 8, shows a flowchart of our suggested "diamond medical knowledge".

Here is the algorithm for the process medical diagnosis bases on diamond intuitionistic fuzzy information.

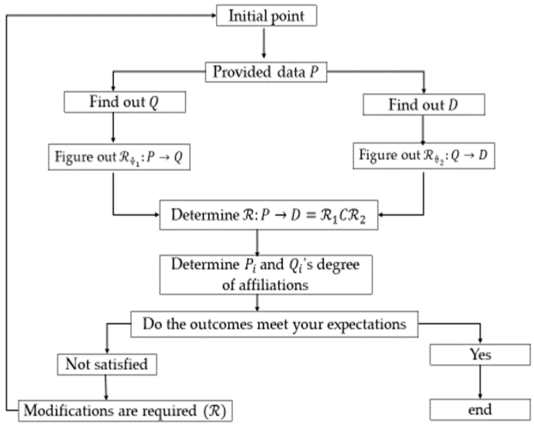

FIGURE 8. Flow chart of diamond medical knowledge.

The following are the primary steps of a typical algorithm for diamond intuitionistic fuzzy information-based medical diagnosis:

1. Identifying the symptoms.
2. Figure out a relationship between $\text{₣}_1 : P \rightarrow Q$.
3. Figure out a relationship between $\text{₣}_2 : Q \rightarrow D$.
4. A composed relation $\text{₣} : P \rightarrow D$.

$$
\begin{aligned}
&\text{₣}_{\vartheta_1} C_{max} \text{₣}_{\vartheta_2} \\
&= \left\{ (\omega, \Lambda),\ k_{\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2}} (\omega, q),\ s_{\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2}} (\omega, q) ; \text{₦}_{\vartheta}\ \ : \text{for all } (\omega, q) \in X \times Z \right\}, \\
&k_{\text{₣}_{\vartheta_1} C_{max} \text{₣}_{\vartheta_2}} (\omega, q) \\
&= \bigvee_{\Lambda} \left\{ k_{\text{₣}_{\vartheta_1}} (\omega, \Lambda) \wedge k_{\text{₣}_{\vartheta_2}} (\Lambda, q) \right\}, \\
&s_{\text{₣}_{\vartheta_1} C_{max} \text{₣}_{\vartheta_2}} (\omega, q) \\
&= \bigwedge_{\Lambda} \left\{ s_{\text{₣}_{\vartheta_1}} (\omega, \Lambda) \vee s_{\text{₣}_{\vartheta_2}} (\Lambda, q) \right\}, \text{₦}_{\vartheta} = \bigvee_{\Lambda} \left\{ \text{₦}_{\vartheta 1} \wedge \text{₦}_{\vartheta 2} \right\}. \\
&\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2} \\
&= min \left\{ \begin{matrix} (\omega, \Lambda),\ k_{\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2}} (\omega, q),\ s_{\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2}} (\omega, q); \\ \text{₦}_{\vartheta}\ \ : \text{for all } (\omega, q) \in X \times Z \end{matrix} \right\} \\
&k_{\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2}} (\omega, q) \\
&= \bigvee_{\Lambda} \left\{ k_{\text{₣}_{\vartheta_1}} (\omega, \Lambda) \wedge k_{\text{₣}_{\vartheta_2}} (\Lambda, q) \right\}, \\
&s_{\text{₣}_{\vartheta_1} C_{min} \text{₣}_{\vartheta_2}} (\omega, q) \\
&= \bigwedge_{\Lambda} \left\{ s_{\text{₣}_{\vartheta_1}} (\omega, \Lambda) \vee s_{\text{₣}_{\vartheta_2}} (\Lambda, q) \right\}, \\
&\text{₦}_{\vartheta} \\
&= \bigwedge_{\Lambda} \left\{ \text{₦}_{\vartheta 1} \vee \text{₦}_{\vartheta 2} \right\}.
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&SC^{*}{}_{\text{₣}} (\omega, q) \\
&= \frac{k_{\text{₣}_{\vartheta}} (\omega, q) + \frac{\text{₦}_{\vartheta}}{2} (2\lambda - 1) + \left(\frac{\text{₦}_{\vartheta}}{2} (2\lambda - 1) - s_{\text{₣}_{\vartheta}} (\omega, q) \right) \pi_{\text{₣}_{\vartheta}} (\omega, q)}{2},
\end{aligned} \tag{14}
$$

TABLE 1. Establish $₣_{\vartheta_1}$ from P to Q.

$₣_{\vartheta_1}$	Q_1	Q_2	Q_3	Q_4	Q_5
P_1	(1,0;.6)	(.5,.5;.5)	(1,0;.6)	(.6,.4;.9)	(0,.8;.8)
P_2	(.9,0;.7)	(.5,.2;.6)	(0,.8;.3)	(.3,.3;.8)	(.5,.4;.9)
P_3	(.4,.4;.9)	(.2,.2;.6)	(.7,.2;.7)	(.3,.4;.5)	(.7,.3;.6)
P_4	(1,.6;.4)	(.5,.5;.2)	(.9,0;.3)	(.3,.6;.9)	(.2,.7;.9)

TABLE 2. Establish $₣_{\vartheta_2}$ from Q to D.

$₣_{\vartheta_2}$	D_1	D_2	D_3	D_4	D_5
Q_1	(1,0;.6)	(.5,.3;.8)	(1,0;.7)	(.3,.7;.9)	(.3,.6;.9)
Q_2	(.7,.2;.9)	(0,.5;.5)	(.3,.6;.8)	(.4,.6;.9)	(.1,.7;.8)
Q_3	(1,0;.9)	(.2,.8;.9)	(.1,0;.5)	(.8,.1;.8)	(.1,.6;.7)
Q_4	(1,0;.9)	(.45,0;.6)	(.9,.1;.9)	(.1,.7;.8)	(.7,.1;.8)
Q_5	(.6,.3;.8)	(.39,.6;.9)	(.49,.5;.9)	(.65,.2;.8)	(.2,.7;.6)

TABLE 3. Evaluating $₣_{\vartheta_{max}} = ₣_{\vartheta_1} c_{max} ₣_{\vartheta_2}$ from P to D.

$₣_{\vartheta_{max}}$	D_1	D_2	D_3	D_4	D_5
P_1	(1,0; .9)	(.5,.3; .8)	(1,0; .9)	(.8,.1; .8)	(.6,.4; .8)
P_2	(.9,0; .8)	(.5,.3; .9)	(.9,0; .9)	(.5,.4; .8)	(.3,.3; .8)
P_3	(.7,.2; .7)	(.4,.4; .8)	(.49,.2; .7)	(.7,.2; .9)	(.3,.4;.9)
P_4	(1,0; .9)	(.5,.5; .9)	(1,0; .9)	(.8,.1; .8)	(.3,.6; .8)

TABLE 4. Evaluating $₣_{\vartheta_{min}} = ₣_{\vartheta_1} c_{min} ₣_{\vartheta_2}$ from P to D.

$₣_{\vartheta_{min}}$	D_1	D_2	D_3	D_4	D_5
P_1	(1,0; .6)	(.5,.3; .5)	(1,0; .6)	(.8,.1; .8)	(.6,.4; .7)
P_2	(.9,0; .7)	(.5,.3; .6)	(.9,0; .5)	(.5,.4; .8)	(.3,.3; .7)
P_3	(.7,.2; .8)	(.4,.4; .6)	(.49,.2; .7)	(.7,.2; .8)	(.3,.4; .6)
P_4	(1,0; .6)	(.5,.5; .5)	(1,0; .5)	(.8,.1; .8)	(.3,.6; .7)

TABLE 5. Characterizing P_i's and D_i's affiliation strength using the formation of Table 3, with decision-making attitude $\lambda = .9$.

$SC^*{}_{₣_{\vartheta_{max}}}$	D_1	D_2	D_3	D_4	D_5
P_1	.59	.316	.59	.483	.38
P_2	.538	.328	.549	.318	.202
P_3	.417	.256	.306	.439	.207
P_4	.59	.34	.55	.483	.208

TABLE 6. Characterizing patient and diagnosis affiliation strength using the formation of Table 4, with decision-making attitude $\lambda = .9$.

$SC^*{}_{₣_{\vartheta_{min}}}$	D_1	D_2	D_3	D_4	D_5
P_1	.56	.28	.56	.483	.37
P_2	.527	.292	.505	.318	.188
P_3	.428	.232	.306	.428	.168
P_4	.56	.3	.55	.483	.197

5. Determine a patient P_i's relationship to a diagnostic D_i's. In Definition 13, utilize $SC^*{}_{₣}$.

Example 1: Let the set of patients $P = \{P_1, P_2, P_3, P_4\}$ is to be diagnosed with respect to set of symptoms $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$, and $D = \{D_1, D_2, D_3, D_4, D_5\}$ is the set of diagnoses.

· Table 1 presents a hypothetical relation $₣_{\vartheta_1} : P \to D$.

· Table 2 presents a hypothetical relation $₣_{\vartheta_2} : D \to Q$.

· Table 3 and Table 4 provide the $max - min - max$ or $max - min - min$ composed relation $₣_{\vartheta} : P \to D$.

· Table 5 and Table 6 compute the degree of association between a patient P_i and a diagnosis D_i.

The relationship between patient P and symptoms Q is shown in Table 1 as $\mathcal{D}{-}IFSs$. It explains the close relationship between patients and symptoms. Similarly, Table 2 shows us the relationship between symptoms Q and diagnostic D. These relationships explain the degree to which symptoms Q require diagnostic D. In a similar vein, Table 3 shows us the relationship between patients Q and diagnostic D

The Tables 5 and 6 present the study of $SC^*{}_{₣_{\vartheta_{max}}}(\omega, ʎ)$ and $SC^*{}_{₣_{\vartheta_{min}}}(\omega, ʎ)$ values, respectively. It is evident that P_1 is a strong patient of both D_1 and D_3 as compare to P_4 and, as evidenced by the equal and highest score values of (P_1, D_1), (P_1, D_3), (P_4, D_1), and (P_4, D_3) and . P_4 is a strong patient of both D_1 and D_3 in significant detail. Although P_3's diagnosis is unknown, but P_3 is partially patient of D_1 and D_4. According to these analyses, there is typically a larger likelihood that a patient has a clear diagnosis if score values are higher than .5. A score of less than **.5** indicates that there is a lower likelihood that patient P will have diagnosis D.

A benefit of $\mathcal{D}{-}IFSs$ is that the medical diagnosis process carried out in the $\mathcal{D} - IFSs$ space cannot be carried out in the FSs, $IFSs$, or interval IFS' environments for one reason because the center of $\mathcal{D} - IFR$ indicates the membership or non-membership of the criterion; the innovative model modifies outcomes based on the lambda value ($\lambda ₦_{\vartheta} IFR$ techniques. This variability underscores the significance of employing $\mathcal{D}{-}IFR$ in $max - min$ models.

TABLE 7. The comparative study of $\mathcal{D}-IFSs$ with fuzzy approaches.

Collections	Remarks	$₦_\vartheta$
FS (Zadeh 1965)	Unable to handle non-membership $s(\omega)$	No
IFS (Atanassov 1986)	just deal with the single value	No
IFS (Atanassov 1989)	cannot deal with the condition value in the shape of diamond	No
CIFS (Atanassov 2020)	cannot deal with the condition value in the shape of diamond	No
$\boldsymbol{\mathcal{D}}$-***IFS***	deal with the condition value in the shape of diamond	yes

B. COMPARATIVE ANALYSIS

These fuzzy sets affect the optimal choice and restricts the decision makers. We provide the novel concept of the $\mathcal{D}-IFSs$, so that decision makers can get better results using this new type of idea.

VI. CONCLUSION

Thought in this article, it can be noticed that our main purpose is to explore the new generalization of IFS which is known as $\mathcal{D}-IFS$. The $\mathcal{D}-IFS$ allows the decision makers to choose the value of membership and non-membership in diamond shapes. During study of $\mathcal{D}-IFS$, we obtained two values λ and $₦_\vartheta$ which represent decision-maker's attitude and uncertainty of the decisions, respectively. This two values give another direction in the field of intuitionistic fuzzy theory. After characterization of some new properties of $\mathcal{D}-IFS$, we obtained novel type of IFS as an exceptional case that is known as $\mathcal{D}-IFS$. Some new basic operations are discussed as well as some model operations are presented. At the end, an application of $\mathcal{D}-IFS$ is provided in evaluations for medical diagnosis by using two novel $max-min$ models. Moreover, some new and classical results have been achieved that can be also considered as application of $\mathcal{D}-IFS$. In future, we will explore this concept for other generalizations of fuzzy sets will try to find some new applications in decision making theory.

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