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Some Fixed Point Theorems for $(a-p)$-Quasicontractions

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Abstract: In this paper, we introduced the notion of $(a-p)$-quasicontraction and proved two generalizations of some classical fixed point theorems. Furthermore, we present some examples to support our results.

Keywords: nonexpansive mappings; approximate fixed point sequence; fixed point; quasicontraction

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1. Introduction

Based probably on ideas of Cauchy and Liouville, Picard [1] developed the method of succesive approximations to prove the existence of solutions of initial value problems for ordinary differential equations. In his famous dissertation from 1922, Banach [2] formulated and proved the Contraction Mapping Principle, which is considered the starting point of fixed point theory. The result was extended and generalized by many researchers in a very dynamical field of research. In what follows, we recall some classical results of this theory.

Theorem 1. *(Edelstein [3]). Let* (Y,δ) *be a compact metric space and let* $T: Y \to Y$ *be a mapping such that* $\delta(Tu,Tv) < \delta(u,v)$ *for all* $u,v \in Y$ *with* $u \neq v$*. Then,* T *has a unique fixed point.*

Theorem 2. *(Hardy-Rogers [4]). Let* (Y,δ) *be a compact metric space and let* $T: Y \to Y$ *be a mapping satisfying inequality*

$$\delta(Tu,Tv) < A\delta(u,v) + B\delta(u,Tu) + C\delta(v,Tv)$$

for all $u,v \in Y$ *and* $u \neq v$*, where* A*,* B*,* C *are positive and* $A+B+C=1$*. Then,* T *has a unique fixed point.*

Theorem 3. *(Greguš [5]). Let* Y *be a Banach space and* C *a closed convex subset of* Y*. Let* $T: Y \to Y$ *be a mapping satisfying inequality*

$$\|Tu - Tv\| \leq a\|u-v\| + b\|u-Tu\| + c\|v-Tv\|$$

for all $u,v \in C$*, where* $0<a<1$*,* $b \geq 0$*,* $c \geq 0$ *and* $a+b+c=1$*. Then,* T *has a unique fixed point.*

Very recently, the authors [6] have introduced the concept of quadratic quasicontraction mapping and proved two generalizations of Theorems 1–3.

Definition 1. *A mapping $T : Y \to Y$ of a metric space Y into itself is said to be a quadratic quasicontraction if there exists $a \in \left(0, \frac{1}{2}\right)$ such that*

$$\delta^2 (Tu, Tv) \le a\delta^2 (u, Tu) + a\delta^2 (v, Tv) + (1 - 2a)\, \delta^2 (u, v) \tag{1}$$

for all $u, v \in X$ and a strict quadratic quasicontraction if in Relation (1) we have the strict inequality for all $u, v \in Y$ and $u \neq v$.

Theorem 4. *(Popescu-Stan [6]) Let (Y, δ) be a compact metric space and let $T : Y \to Y$ be a strict quadratic quasicontraction. Then, T has a unique fixed point $w \in Y$. Moreover, if T is continuous, then, for each $u \in Y$, the sequence of iterates $\{T^n u\}$ converges to w.*

Theorem 5. *(Popescu-Stan [6]) Let Y be a Banach space and C be a closed convex subset of Y. Let $T : C \to C$ be a mapping satisfying inequality*

$$\|Tu - Tv\|^2 \le a \|u - Tu\|^2 + a \|v - Tv\|^2 + b \|u - v\|^2$$

for all $u, v \in C$, where $0 < a < \frac{1}{2}$, $b = 1 - 2a$. Then, T has a unique fixed point.

In this paper, we extend the notion of quadratic quasicontraction mapping and prove the analogues of Theorems 4 and 5 for a new type of mappings.

The following lemma will be necessary in the proof of the main result:

Lemma 1. *(see Lemma 1.6 [7]) Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $0 \le k < 1$, so that*

$$\alpha_{n+1} \le k\alpha_n + \beta_n,$$

for all $n \ge 0$. If $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main Results

Definition 2. *Let $p \in \mathbb{N}$ and $a \in \left(0, \frac{1}{2}\right)$. A mapping $T : Y \to Y$ of a metric space Y into itself is said to be an $(a - p)$-quasicontraction if*

$$\delta^p (Tu, Tv) \le a\delta^p (u, Tu) + a\delta^p (v, Tv) + (1 - 2a)\, \delta^p (u, v) \tag{2}$$

for all $u, v \in Y$ and a strict $(a - p)$-quasicontraction if we have strict inequality in Relation (2) for all $u, v \in Y$ with $u \neq v$.

Remark 1. *For $p = 2$ we have the notion of quadratic quasicontraction mapping.*

Proposition 1. *If $p, q \in \mathbb{N}$, $p \le q$, then every $(a - p)$-quasicontraction is an $(a - q)$-quasicontraction.*

Proof. Suppose T is an $(a - p)$-quasicontraction. Using the convexity of $t \to t^{\frac{q}{p}}$, we have

$$\begin{aligned} \delta^q (Tu, Tv) &= (\delta^p (Tu, Tv))^{\frac{q}{p}} \\ &\le (a\delta^p (u, Tu) + a\delta^p (v, Tv) + (1 - 2a)\, \delta^p (u, v))^{\frac{q}{p}} \\ &\le a\delta^q (u, Tu) + a\delta^q (v, Tv) + (1 - 2a)\, \delta^q (u, v) \end{aligned}$$

for all $u, v \in Y$. Hence T is an $(a - q)$-quasicontraction. □

Inspired by Example 3.3 ([8]) we give an example which shows that not every $(a-q)$-quasicontraction is an $(a-p)$-quasicontraction if $p<q$, where $a \neq \frac{1}{3}$.

Example 1. *Let* $c_1 := \left(\frac{1-2a}{a}\right)^{\frac{1}{q-1}} > 0$ *and* $c_2 := \frac{1-2a}{(1+c_1)^{q-1}} \in (0,1)$. *Let* τ *be the unique real number satisfying*

$$0<\tau<1 \text{ and } c_2 + a\left(1-\tau\right)^q = \tau^q. \tag{3}$$

Let $Y=[0,1]$, $\delta\left(u,v\right)=|u-v|$ *and* $T:Y\to Y$ *such that*

$$Tu = \begin{cases} 0 & \text{if } u\in[0,1) \\ \tau & \text{if } u=1 \end{cases}.$$

Then, T is an $(a-q)$*-quasicontraction, but T is not an* $(a-p)$*-quasicontraction for any* $p\in\mathbb{N}$ *with* $p<q$.

Proof. Let $f:[0,1]\to\mathbb{R}$ defined by

$$f\left(t\right) = c_2 + a\left(1-t\right)^q - t^q.$$

Obviously, f is continuous on $[0,1]$ and differentiable on $(0,1)$. We have

$$f'\left(t\right) = -aq\left(1-t\right)^{q-1} - qt^{q-1} < 0$$

for any $t\in(0,1)$, so f is strictly decreasing. Since $f\left(0\right)=c_2+a>0$ and $f\left(1\right)=c_2-1<0$, we note that there exists a unique real number $\tau\in(0,1)$ satisfying Relation (3). Now, let g be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$g\left(u\right) = au^q + \left(1-2a\right)\left(1-u\right)^q + a\left(1-\tau\right)^q.$$

Then, g is continuous on $[0,1]$ and differentiable on $(0,1)$. We have

$$g'\left(u\right) = aqu^{q-1} - \left(1-2a\right)q\left(1-u\right)^{q-1},$$

so $g'\left(u\right)=0 \Rightarrow \left(\frac{u}{1-u}\right)^{q-1} = \frac{1-2a}{a} \Rightarrow \frac{u}{1-u} = \left(\frac{1-2a}{a}\right)^{\frac{1}{q-1}} = c_1 \Rightarrow u = \frac{c_1}{1+c_1}$.

Putting $v := \frac{c_1}{1+c_1} \in (0,1)$, we obtain

$$\begin{array}{lcl} g'\left(u\right) & < & 0 \text{ if } u<v \text{ and} \\ g'\left(u\right) & > & 0 \text{ if } u>v. \end{array}$$

Hence

$$\min\left\{g\left(u\right): 0\le u\le 1\right\} = g\left(v\right),$$

where

$$\begin{aligned} g\left(v\right) &= a\left(\frac{c_1}{1+c_1}\right)^q + \left(1-2a\right)\frac{1}{\left(1+c_1\right)^q} + a\left(1-\tau\right)^q \\ &= a\frac{c_1^q}{1+c_1}\cdot\frac{1}{\left(1+c_1\right)^{q-1}} + \frac{1}{1+c_1}\cdot\frac{1-2a}{\left(1+c_1\right)^{q-1}} + a\left(1-\tau\right)^q \\ &= a\frac{c_1^q}{1+c_1}\cdot\frac{c_2}{1-2a} + \frac{c_2}{1+c_1} + a\left(1-\tau\right)^q \\ &= \frac{c_1c_2}{1+c_1} + \frac{c_2}{1+c_1} + a\left(1-\tau\right)^q \\ &= c_2 + a\left(1-\tau\right)^q \\ &= \tau^q. \end{aligned}$$

For $u < 1$, we have

$$\begin{aligned}
\delta^q(Tu, T1) &= \tau^q = g(v) \\
&\leq g(u) \\
&= au^q + (1-2a)(1-u)^q + a(1-\tau)^q \\
&= a\delta^q(u, Tu) + a\delta^q(1, T1) + (1-2a)\delta^q(u, 1).
\end{aligned}$$

Therefore, T is an $(a-q)$-quasicontraction.

Since $a \neq \frac{1}{3}$ we have $c_1 \neq 1$, $v \neq \frac{1}{2}$ and $v \neq 1 - v$. Using the strict convexity of $t \to t^{\frac{q}{p}}$, we get

$$\begin{aligned}
& a\delta^p(v, Tv) + a\delta^p(1, T1) + (1-2a)\delta^p(v, 1) \\
= & av^p + (1-2a)(1-v)^p + a(1-\tau)^p \\
= & \left(av^p + (1-2a)(1-v)^p + a(1-\tau)^p\right)^{\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)} \\
< & \left(av^q + (1-2a)(1-v)^q + a(1-\tau)^q\right)^{\frac{p}{q}} \\
= & \left(\tau^q\right)^{\frac{p}{q}} = \tau^p = \delta^p(Tv, T1).
\end{aligned}$$

Hence, T is not an $(a-p)$-quasicontraction. □

The folowing example shows that for $a = \frac{1}{3}$ not every $(a-p)$-quasicontraction $(p \geq 3)$ is a quadratic quasicontraction.

Example 2. *(see Example 1 [6]) Let* $Y = [-1, 1]$, $\delta(u, v) = |u - v|$ *and*

$$Tu = \begin{cases} 0, & u \in \left[-1, \frac{1}{2}\right], \\ -1, & u \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Then, T satisfies Inequality (2) with $a = \frac{1}{3}$ *and* $p \geq 3$, *but T is not a quadratic quasicontraction.*

Proof. Let

$$E_p(u, v) = a\delta^p(u, Tu) + a\delta^p(v, Tv) + (1-2a)\delta^p(u, v).$$

For $u \in \left[-1, \frac{1}{2}\right]$ and $v \in \left(\frac{1}{2}, 1\right]$ we have $\delta(Tu, Tv) = 1$ and

$$\begin{aligned}
E_p(u, v) &= \frac{1}{3}\delta^p(u, Tu) + \frac{1}{3}\delta^p(v, Tv) + \frac{1}{3}\delta^p(u, v) \\
&= \frac{1}{3}|u|^p + \frac{1}{3}(v+1)^p + \frac{1}{3}(v-u)^p \\
&> \frac{1}{3}\left(\frac{1}{2} + 1\right)^p = \frac{3^{p-1}}{2^p} > 1, \text{ because } p \geq 3.
\end{aligned}$$

Therefore, T is an $(\frac{1}{3} - p)$-quasicontraction with $p \geq 3$.

For $u = 0$ and $v = \frac{3}{5}$ we have $\delta\left(T0, T\frac{3}{5}\right) = \delta(0, -1) = 1$ and

$$\begin{aligned}
E_2\left(0, \frac{3}{5}\right) &= \frac{1}{3}\delta^2(0, T0) + \frac{1}{3}\delta^2\left(\frac{3}{5}, T\frac{3}{5}\right) + \frac{1}{3}\delta^2\left(0, \frac{3}{5}\right) \\
&= \frac{1}{3} \cdot \frac{64}{25} + \frac{1}{3} \cdot \frac{9}{25} = \frac{73}{75} < \delta\left(T0, T\frac{3}{5}\right),
\end{aligned}$$

so T is not a quadratic quasicontraction with $a = \frac{1}{3}$.

Moreover, since

$$\begin{aligned} E_1\left(0,\frac{3}{5}\right) &= a\delta\left(0,T0\right)+a\delta\left(\frac{3}{5},T\frac{3}{5}\right)+\left(1-2a\right)\delta\left(0,\frac{3}{5}\right) \\ &= a\cdot\frac{8}{5}+\left(1-2a\right)\cdot\frac{3}{5}=\frac{2a+3}{5}<1, \end{aligned}$$

we get that T is not an $(a-1)$-quasicontraction for any $a\in\left(0,\frac{1}{2}\right)$. $\square$

The following theorem is a generalization of Theorem 4 and implicitly a generalization of Theorems 1 and 2.

Theorem 6. *Let (Y,δ) be a compact metric space and let $T:Y\to Y$ be a strict $(a-p)$-quasicontraction. Then, T has a unique fixed point $w\in Y$. Moreover, if T is continuous, then, for each $u\in Y$, the sequence of iterates $\{T^n u\}$ converges to w.*

Proof. Letting $v=Tu$ in Inequality (2), we have for all $u\in Y$ with $u\neq Tu$

$$\delta^p\left(Tu,T^2u\right)<a\delta^p\left(u,Tu\right)+a\delta^p\left(Tu,T^2u\right)+\left(1-2a\right)\delta^p\left(u,Tu\right).$$

This implies $\delta\left(Tu,T^2u\right)<\delta\left(u,Tu\right)$.

Now, let $\beta=\inf\{\delta\left(u,Tu\right):u\in Y\}$. Since Y is compact there exists a sequence $\{u_n\}\subset Y$ such that $u_n\to w\in Y$, $Tu_n\to z$ and $\beta=\lim\limits_{n\to\infty}\delta\left(u_n,Tu_n\right)=\delta\left(w,z\right)$.

If there exists a subsequence $\left\{u_{n(k)}\right\}$ of $\{u_n\}$ such that $u_{n(k)}=z$ for every $k\in\mathbb{N}$, then we get $w=z=Tz$, so T has a fixed point. Otherwise, we suppose that there exists $N\in\mathbb{N}$ such that $u_n\neq z$ for all $n\geq N$. Taking $u=u_n$ and $v=z$ in Inequality (2), we obtain

$$\delta^p\left(Tu_n,Tz\right)<a\delta^p\left(u_n,Tu_n\right)+a\delta^p\left(z,Tz\right)+\left(1-2a\right)\delta^p\left(u_n,z\right).$$

Letting $n\to\infty$, we get

$$\delta^p\left(z,Tz\right)\leq a\delta^p\left(w,z\right)+a\delta^p\left(z,Tz\right)+\left(1-2a\right)\delta^p\left(w,z\right).$$

Thus, $\delta\left(z,Tz\right)\leq\delta\left(w,z\right)=\beta$. By definition of β, we obtain $\delta\left(z,Tz\right)=\beta$.

If $\beta>0$, since $\delta\left(Tz,T^2z\right)<\delta\left(z,Tz\right)=\beta$, we have a contradiction. Therefore, we get $\beta=0$ and so $w=z=Tz$, i.e., z is a fixed point of T.

If t is another fixed point of T, by Inequality (2), taking $u=z$ and $v=t$, we obtain

$$\delta^p\left(Tz,Tt\right)\leq a\delta^p\left(z,Tz\right)+a\delta^p\left(t,Tt\right)+\left(1-2a\right)\delta^p\left(z,t\right),$$

by where

$$\delta^p\left(z,t\right)<\left(1-2a\right)\delta^p\left(z,t\right),$$

which is a contradiction.

Now assume T is continuous. Let $u_0\in Y$ and define a sequence $\{u_n\}\subset Y$ by $u_n:=Tu_{n-1}=T^nu_0$. Suppose there exists $N\in\mathbb{N}\cup\{0\}$ such that $u_N=z$. Then, we have $u_n=z$ for all $n\geq N$, so $u_n\to z$. Otherwise, we suppose $u_n\neq z$ for all $n\geq N$. By uniqueness of z, we get $u_n\neq u_{n+1}$ for every $n\in\mathbb{N}\cup\{0\}$. Hence, we have $\delta\left(u_n,u_{n+1}\right)=\delta\left(Tu_{n-1},Tu_n\right)<\delta\left(u_{n-1},u_n\right)$ for every $n\in\mathbb{N}$, so the sequence $\{\delta\left(u_n,u_{n+1}\right)\}_{n\in\mathbb{N}}$ is decreasing and positive. Therefore, there exists $b=\lim\limits_{n\to\infty}\delta\left(u_{n-1},u_n\right)$. We claim that $b=0$. Since Y is compact, there exists a subsequence $\left\{u_{n(k)}\right\}$ of $\{u_n\}$ such that $u_{n(k)}\to z\in Y$ as $k\to\infty$. If $b>0$, we have

$$b = \lim_{k\to\infty} \delta\left(u_{n(k)}, u_{n(k)+1}\right) = \delta(z, Tz) > 0,$$

$$b = \lim_{k\to\infty} \delta\left(u_{n(k)+1}, u_{n(k)+2}\right) = \delta\left(Tz, T^2 z\right) > 0.$$

Hence, we get $\delta(z, Tz) = \delta\left(Tz, T^2z\right) = b > 0$, which is a contradiction. Thus, $b = 0$. Now, taking $u = u_n$ and $v = z$ in Inequality (2), we obtain

$$\delta^p(u_{n+1}, z) = \delta^p(Tu_n, z) < a\delta^p(u_n, Tu_n) + a\delta^p(z, Tz) + (1-2a)\,\delta^p(u_n, z).$$

This implies

$$\alpha_{n+1} < (1-2a)\,\alpha_n + \beta_n,$$

where $\alpha_n := \delta^p(x_n, z)$ and $\beta_n = a\delta^p(u_n, u_{n+1})$. Since $\lim\limits_{n\to\infty}\beta_n = 0$, by Lemma 1, we get $\lim\limits_{n\to\infty}\alpha_n = 0$, i.e., $\lim\limits_{n\to\infty} u_n = z$. □

Example 3. *Let* $Y = [-2, 2]$, $\delta(u, v) = |u - v|$, $T : Y \to Y$ *defined by*

$$Tu = \begin{cases} \frac{1-u}{2}, & \text{if } u \in [-2, -1) \\ -\frac{2}{5}, & \text{if } u = -1 \\ 0, & \text{if } u \in (-1, 1) \\ \frac{2}{5}, & \text{if } u = 1 \\ \frac{-1-u}{2} & \text{if } u \in (1, 2] \end{cases}.$$

Then, T *is an* $(\frac{1}{3} - p)$*-quasicontraction for* $p \geq 4$*, but* T *is not an* $(\frac{1}{3} - 2)$*-quasicontraction (quadratic quasicontraction). Moreover,* T *is not asymptotic regular.*

Proof. We distinguish 13 cases:

1^0 If $u, v \in [-2, -1)$, $u < v$, we have:

$$\delta(Tu, Tv) = \frac{v-u}{2},$$

$$E_1(u, v) = \frac{1-3u}{6} + \frac{1-3v}{6} + \frac{v-u}{3} = \frac{2-5u-v}{6},$$
$$\delta(Tu, Tv) < E_1(u, v) \Longleftrightarrow 2v + u < 1,$$

which is obvious.

2^0 By symmetry, the case $u, v \in (1, 2]$, $u < v$ is similar.

3^0 If $u \in [-2, -1)$, $v \in (-1, 1)$, we have:

$$\delta(Tu, Tv) = \frac{1-u}{2},$$

$$E_1(u, v) = \frac{1-3u}{6} + \frac{|v|}{3} + \frac{v-u}{3} = \frac{1-5u+2v+2|v|}{6},$$
$$\delta(Tu, Tv) < E_1(u, v) \Longleftrightarrow 2 + 2u < 2v + 2|v|,$$

which is obvious $(2 + 2u < 0 \leq 2v + 2|v|)$.

4^0 By symmetry, the case $u \in (1, 2]$, $v \in (-1, 1)$ is similar.

5^0 If $u \in [-2, -1)$, $v \in (1, 2]$, we have:

$$\delta(Tu, Tv) = \frac{2-u+v}{2},$$

$$E_1(u,v) = \frac{1-3u}{6} + \frac{1+3v}{6} + \frac{v-u}{3} = \frac{2-5u+5v}{6},$$
$$\delta(Tu,Tv) < E_1(u,v) \Longleftrightarrow 2 < v-u,$$

which is obvious.

6^0 If $u, v \in (-1,1)$, $u \neq v$, we have:

$$\delta(Tu,Tv) = 0 < \frac{|u|}{3} + \frac{|v|}{3} + \frac{|u-v|}{3} = E_1(u,v).$$

7^0 If $u = -1$, $v = 1$, we have:

$$\delta(Tu,Tv) = \frac{4}{5},$$
$$E_1(u,v) = \frac{1}{3}\cdot\frac{3}{5} + \frac{1}{3}\cdot\frac{3}{5} + \frac{1}{3}\cdot 2 = \frac{16}{15}$$
$$\delta(Tu,Tv) < E_1(u,v)$$

8^0 For $u = 1$, $v \in (-1,1)$, we get:

$$\delta(Tu,Tv) = \frac{2}{5},$$
$$E_1(u,v) = \frac{1}{3}\cdot\frac{3}{5} + \frac{|v|}{3} + \frac{1-v}{3} = \frac{8+5|v|-5v}{15}$$
$$\delta(Tu,Tv) < E_1(u,v) \Longleftrightarrow 0 < 2+5|v|-5v,$$

which is obvious.

9^0 By symmetry, the case $u = -1$, $v \in (-1,1)$ is similar.

10^0 For $u = 1$, $v \in [-2,-1)$, we have:

$$\delta(Tu,Tv) = \frac{1-5v}{10},$$
$$E_1(u,v) = \frac{1}{3}\cdot\frac{1-3v}{2} + \frac{1}{3}\cdot\frac{3}{5} + \frac{1-v}{3} = \frac{21-25v}{30}$$
$$\delta(Tu,Tv) < E_1(u,v) \Longleftrightarrow 10v < 18,$$

which is obvious.

11^0 By symmetry, the case $u = -1$, $v \in (1,2]$ is similar.

12^0 For $u = 1$, $v \in (1,2]$, we have:

$$\delta^p(Tu,Tv) = \left(\frac{9+5v}{10}\right)^p,$$
$$E_p(u,v) = \frac{1}{3}\cdot\left(\frac{3}{5}\right)^p + \frac{1}{3}\cdot\left(\frac{3v+1}{2}\right)^p + \frac{1}{3}\cdot(1-v)^p$$

Since the function $v \to \frac{15v+5}{9+5v}$ is increasing and $p \geq 4$, we obtain

$$\left(\frac{15v+5}{9+5v}\right)^p > \left(\frac{15+5}{9+5}\right)^p = \left(\frac{10}{7}\right)^p \geq \left(\frac{10}{7}\right)^4 \geq 3,$$

so

$$\left(\frac{9+5v}{10}\right)^p < \frac{1}{3}\cdot\left(\frac{3v+1}{2}\right)^p,$$

by where $\delta^p(Tu,Tv) < E_p(u,v)$.

13^0 By symmetry, the case $u = -1$, $v \in [-2,-1)$ is similar.

By Proposition 1 is obvious that $\delta\left(Tu,Tv\right) < E_1\left(u,v\right) \Rightarrow \delta^p\left(Tu,Tv\right) < E_p\left(u,v\right)$. Therefore we get that in all cases we have $\delta^p\left(Tu,Tv\right) < E_p\left(u,v\right)$. Hence, T is an $(\frac{1}{3}-p)$-quasicontraction for $p \geq 4$.

If $u = 1, v = \frac{4}{3}$, we have:

$$\delta^2\left(Tu,Tv\right) = \frac{6627}{2700},$$

$$E_2\left(u,v\right) = \frac{1}{3}\cdot\frac{9}{25} + \frac{1}{3}\cdot\frac{25}{4} + \frac{1}{3}\cdot\frac{1}{9} = \frac{5949}{2700},$$

so $\delta^2\left(Tu,Tv\right) > E_2\left(u,v\right)$. This implies that T is not a quadratic quasicontraction.

It is easy to prove that $T^n u = (-1)^n \cdot \frac{u+2^n-1}{2^n}$ for $u \in (1,2]$, so $T^n 2 = (-1)^n \cdot \frac{2^n+1}{2^n}$, for all $n \in \mathbb{N}$, and $\delta\left(T^n 2, T^{n+1} 2\right) = \frac{2^n+1}{2^n} + \frac{2^{n+1}+1}{2^{n+1}} > 2$ i.e., T is not asymptotic regular. □

The following lemma plays a very important role in the next theorem.

Lemma 2. *Let C be a nonempty closed subset of a complete metric space (Y,δ) and let $T : C \to C$ be an $(a-p)$-quasicontraction mapping. Assume that there exist constants $c, b \in \mathbb{R}$ such that $0 \leq c < 1$ and $b > 0$. If for every $u \in C$ there exists $v \in C$ such that $\delta\left(v,Tv\right) \leq c\delta\left(u,Tu\right)$ and $\delta\left(v,u\right) \leq b\delta\left(u,Tu\right)$, then T has a unique fixed point.*

Proof. Let $u_0 \in C$. We construct the sequence $\{u_n\} \subset C$ such that

$$\delta\left(u_{n+1},Tu_{n+1}\right) \leq c\delta\left(u_n,Tu_n\right),$$

$$\delta\left(u_{n+1},u_n\right) \leq b\delta\left(u_n,Tu_n\right), n = 0,1,2,...$$

Since

$$\delta\left(u_{n+1},u_n\right) \leq b\delta\left(u_n,Tu_n\right) \leq bc\delta\left(u_{n-1},Tu_{n-1}\right) \leq ... \leq bc^n\delta\left(u_0,Tu_0\right)$$

it is easy to prove that $\{u_n\}$ is a Cauchy sequence. By completeness of C, there exists $w \in C$ such that $\lim\limits_{n\to\infty} u_n = w$. From the above inequalities and sandwich theorem, we get $\lim\limits_{n\to\infty}\delta\left(u_n,Tu_n\right) = 0$, so $\lim\limits_{n\to\infty} Tu_n = w$ and

$$\delta^p\left(Tu_n,Tw\right) \leq a\delta^p\left(u_n,Tu_n\right) + a\delta^p\left(w,Tw\right) + (1-2a)\,\delta^p\left(u_n,w\right).$$

Letting $n \to \infty$, we obtain

$$\delta^p\left(w,Tw\right) \leq a\delta^p\left(w,Tw\right).$$

This implies $\delta\left(w,Tw\right) = 0$, i.e., $w = Tw$.

If t is another fixed point of T, taking $u = w, v = t$ in Inequality (2) we obtain

$$\delta^p\left(Tw,Tt\right) \leq a\delta^p\left(w,Tw\right) + a\delta^p\left(t,Tt\right) + (1-2a)\,\delta^p\left(w,t\right),$$

hence

$$\delta^p\left(w,t\right) \leq (1-2a)\,\delta^p\left(w,t\right).$$

Therefore, $\delta\left(w,t\right) = 0$, which is a contradiction. □

The next theorem is a partial generalization of Theorem 5.

Theorem 7. *Let Y be a Banach space and C be a closed convex subset of X. Let $T : C \to C$ be a mapping satisfying the inequality:*

$$\|Tu - Tv\|^p \leq a\,\|u - Tu\|^p + a\,\|v - Tv\|^p + b\,\|u - v\|^p \tag{4}$$

for all $u, v \in C$*, where* $0 < a < \frac{1}{2}$*,* $b = 1 - 2a$*,* $b \leq \frac{\left(\sqrt[p]{2^{p+1}-1}-1\right)^p - 1}{2^p - 1}$*,* $p \in \mathbb{N}$*,* $p \geq 2$*. Then,* T *has a unique fixed point.*

Proof. Taking $v = Tu$ in (4), we have:

$$\left\|Tu - T^2u\right\|^p \leq a\left\|u - Tu\right\|^p + a\left\|Tu - T^2u\right\|^p + b\left\|u - Tu\right\|^p .$$

This implies

$$\left\|Tu - T^2u\right\| \leq \left\|u - Tu\right\|, \text{ for all } u \in C. \tag{5}$$

Now, let $u \in C$ arbitrary fixed and $z = \frac{1}{2}T^2u + \frac{1}{2}T^3u$. Since C is convex, we have $z \in C$. Then, by Inequalities (4) and (5), we get

$$\begin{aligned} \left\|Tu - T^3u\right\|^p &\leq a\left\|u - Tu\right\|^p + a\left\|T^2u - T^3u\right\|^p + b\left\|u - T^2u\right\|^p \\ &\leq 2a\left\|u - Tu\right\|^p + b\left(\left\|u - Tu\right\| + \left\|Tu - T^2u\right\|\right)^p \\ &\leq (2a + 2^p b)\left\|u - Tu\right\|^p , \end{aligned}$$

so

$$\left\|Tu - T^3u\right\| \leq \sqrt[p]{2a + 2^p b}\left\|u - Tu\right\| . \tag{6}$$

Therefore,

$$\begin{aligned} \left\|Tu - z\right\| &= \left\|\frac{1}{2}\left(Tu - T^2u\right) + \frac{1}{2}\left(Tu - T^3u\right)\right\| \\ &\leq \frac{1}{2}\left\|Tu - T^2u\right\| + \frac{1}{2}\left\|Tu - T^3u\right\| \\ &\leq \frac{1}{2}\left\|u - Tu\right\| + \frac{1}{2}\sqrt[p]{2a + 2^p b}\left\|u - Tu\right\| \\ &= \frac{1}{2}\left(1 + \sqrt[p]{2a + 2^p b}\right)\left\|u - Tu\right\| . \end{aligned} \tag{7}$$

In addition,

$$\left\|T^2u - z\right\| = \frac{1}{2}\left\|T^2u - T^3u\right\| \leq \frac{1}{2}\left\|u - Tu\right\| . \tag{8}$$

Now, by Inequalities (4), (5) and (7), we obtain

$$\begin{aligned} &\left\|T^2u - Tz\right\|^p \\ \leq\; & a\left\|Tu - T^2u\right\|^p + a\left\|z - Tz\right\|^p + b\left\|Tu - z\right\|^p \\ \leq\; & a\left\|u - Tu\right\|^p + a\left\|z - Tz\right\|^p + b\left(\frac{1 + \sqrt[p]{2a + 2^p b}}{2}\right)^p \left\|u - Tu\right\|^p \\ =\; & a\left\|z - Tz\right\|^p + \left[a + b\left(\frac{1 + \sqrt[p]{2a + 2^p b}}{2}\right)^p\right]\left\|u - Tu\right\|^p . \end{aligned} \tag{9}$$

By Inequalities (4), (5) and (8), we obtain

$$\begin{aligned}\left\|T^3u-Tz\right\|^p &\leq a\left\|T^2u-T^3u\right\|^p+a\left\|z-Tz\right\|^p+b\left\|T^2u-z\right\|^p\\ &\leq a\left\|u-Tu\right\|^p+a\left\|z-Tz\right\|^p+\frac{b}{2^p}\left\|u-Tu\right\|^p\\ &= a\left\|z-Tz\right\|^p+\left(a+\frac{b}{2^p}\right)\left\|u-Tu\right\|^p. \qquad (10)\end{aligned}$$

Since

$$\begin{aligned}\left\|z-Tz\right\| &= \left\|\frac{1}{2}\left(T^2u-Tz\right)+\frac{1}{2}\left(T^3u-Tz\right)\right\|\\ &\leq \frac{1}{2}\left\|T^2u-Tz\right\|+\frac{1}{2}\left\|T^3u-Tz\right\|,\end{aligned}$$

by Inequalities (9) and (10), we get

$$\begin{aligned}\left\|z-Tz\right\| &\leq \frac{1}{2}\left\{a\left\|z-Tz\right\|^p+\left[a+b\left(\frac{1+\sqrt[p]{2a+2^pb}}{2}\right)^p\right]\left\|u-Tu\right\|^p\right\}^{\frac{1}{p}}\\ &\quad+\frac{1}{2}\left[a\left\|z-Tz\right\|^p+\left(a+\frac{b}{2^p}\right)\left\|u-Tu\right\|^p\right]^{\frac{1}{p}}. \qquad (11)\end{aligned}$$

If $u=Tu$, then u is a fixed point of T. Otherwise, dividing Inequality (11) by $\frac{1}{2}\left\|u-Tu\right\|$, we get

$$\begin{aligned}2\cdot\frac{\left\|z-Tz\right\|}{\left\|u-Tu\right\|} &\leq \left[a\frac{\left\|z-Tz\right\|^p}{\left\|u-Tu\right\|^p}+a+b\left(\frac{1+\sqrt[p]{2a+2^pb}}{2}\right)^p\right]^{\frac{1}{p}}\\ &\quad+\left[a\frac{\left\|z-Tz\right\|^p}{\left\|u-Tu\right\|^p}+a+\frac{b}{2^p}\right]^{\frac{1}{p}}.\end{aligned}$$

Denoting $t:=\frac{\left\|z-Tz\right\|^p}{\left\|u-Tu\right\|^p}$, we obtain

$$2t^{\frac{1}{p}}\leq\left[at+a+b\left(\frac{1+\sqrt[p]{2a+2^pb}}{2}\right)^p\right]^{\frac{1}{p}}+\left[at+a+\frac{b}{2^p}\right]^{\frac{1}{p}},$$

which implies $z=Tz$ or $f\left(t\right)\geq 2$, where

$$f\left(t\right)=\left[a+\frac{a}{t}+\frac{b}{t}\left(\frac{1+\sqrt[p]{2a+2^pb}}{2}\right)^p\right]^{\frac{1}{p}}+\left(a+\frac{a}{t}+\frac{b}{2^p\cdot t}\right)^{\frac{1}{p}}$$

Clearly, f is a decreasing function and

$$\begin{aligned}f\left(1\right) &= \left[2a+b\left(\frac{1+\sqrt[p]{2a+2^pb}}{2}\right)^p\right]^{\frac{1}{p}}+\left(a+\frac{b}{2^p}\right)^{\frac{1}{p}}\\ &= \left[1-b+b\left(\frac{1+\sqrt[p]{1-b+2^pb}}{2}\right)^p\right]^{\frac{1}{p}}+\left(1-b+\frac{b}{2^p}\right)^{\frac{1}{p}}.\end{aligned}$$

We claim that $f\left(1\right)<2$. Since $b\leq\frac{\left(\sqrt[p]{2^{p+1}-1}-1\right)^p-1}{2^p-1}$, we have $\left(\frac{1+\sqrt[p]{1-b+2^pb}}{2}\right)^p\leq\frac{2^{p+1}-1}{2^p}\leq 2-\frac{1}{2^p}$ and then

$$\begin{aligned} f(1) &< \left[1-b+b\left(2-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}+\left(1-b+\frac{b}{2^p}\right)^{\frac{1}{p}} \\ &= \left(1+b-\frac{b}{2^p}\right)^{\frac{1}{p}}+\left(1-b+\frac{b}{2^p}\right)^{\frac{1}{p}}. \end{aligned}$$

Using strict concavity of $t \to t^{\frac{1}{p}}$, we obtain

$$f(1) < 2\left(\frac{1+b-\frac{b}{2^p}+1-b+\frac{b}{2^p}}{2}\right)^{\frac{1}{p}} = 2.$$

Since f is a decreasing function and $f(t) \geq 2$, there exists $c < 1$ such that $t \leq c$. Therefore, $\|z - Tz\| \leq \sqrt{c}\,\|u - Tu\|$.

Now, since

$$\begin{aligned} \|z-u\| &\leq \frac{1}{2}\left\|T^2u-u\right\|+\frac{1}{2}\left\|T^3u-u\right\| \\ &\leq \frac{1}{2}\left(\left\|T^2u-Tu\right\|+\|Tu-u\|\right) \\ &\quad +\frac{1}{2}\left(\left\|T^3u-T^2u\right\|+\left\|T^2u-Tu\right\|+\|Tu-u\|\right) \\ &\leq \frac{5}{2}\|Tu-u\|, \end{aligned}$$

applying Lemma 2, we get that T has a unique fixed point. □

Example 4. *Let* $Y = l^\infty(\mathbb{R})$ *be the set of bounded sequence of real numbers and* $\|u\| = \sup_{n\in\mathbb{N}} |u_n|$, *where* $u = \{u_n\}_{n\in\mathbb{N}}$. *It is well known that* $(Y, \|\cdot\|)$ *is a Banach space. Let* $C = \{u \in Y : \|u\| \leq 1\}$ *and* $T : C \to C$ *defined by*

$$Tu = \begin{cases} \frac{\mathbf{1}}{\mathbf{3}}, & \text{if } u = -\mathbf{1} \\ -\mathbf{1}, & \text{if } u_n \in \left[\frac{1}{3}, 1\right] \text{ for every } n \in \mathbb{N} \\ \mathbf{0}, & \text{otherwise} \end{cases},$$

where $u = \{u_n\}_{n\in\mathbb{N}}$, $\mathbf{c} = \{c, c, c, \ldots\}$. *Clearly, C is closed, convex and not compact. Since* $T^n(-\mathbf{1}) = \frac{\mathbf{1}}{\mathbf{3}}$ *if n is odd and* $T^n(-\mathbf{1}) = -\mathbf{1}$ *if n is even, we note that T is not asymptotic regular.*

If $u = -\mathbf{1}$ *and* $v = \{v_n\}_{n\in\mathbb{N}}$, *where* $v_n \in \left[\frac{1}{3}, 1\right]$ *for every* $n \in \mathbb{N}$, *then* $\|Tu - Tv\| = \frac{4}{3}$ *and*

$$\begin{aligned} E_1(u,v) &= \frac{4}{3}a + a\sup_{n\in\mathbb{N}}(1+v_n) + b\sup_{n\in\mathbb{N}}(1+v_n) \\ &\geq \frac{4}{3}a + \frac{4}{3}(a+b) = \frac{4}{3}. \end{aligned}$$

Hence $\|Tu - Tv\| \leq E_1(u,v)$, *and then* $\|Tu - Tv\|^3 \leq E_3(u,v)$.

If $u = -\mathbf{1}$ *and* $v \neq u$, $v = \{v_n\}_{n\in\mathbb{N}}$, *where there exists* n_0 *such that* $v_{n_0} \notin \left[\frac{1}{3}, 1\right]$, *then* $\|Tu - Tv\| = \frac{1}{3}$ *and*

$$\begin{aligned} E_1(u,v) &= \frac{4}{3}a + a\sup_{n\in\mathbb{N}}|v_n| + b\sup_{n\in\mathbb{N}}(1+v_n) \\ &\geq \frac{4}{3}a \geq \frac{1}{3} \text{ if } a \geq \frac{1}{4}. \end{aligned}$$

If $u = \{u_n\}_{n\in\mathbb{N}}$ *where* $u_n \in \left[\frac{1}{3}, 1\right]$ *for every* $n \in \mathbb{N}$ *and* $v = \{v_n\}_{n\in\mathbb{N}}$, *where there exists* n_0 *such that* $v_{n_0} \notin \left[\frac{1}{3}, 1\right]$, *then* $\|Tu - Tv\| = 1$ *and*

$$\begin{aligned} E_3(u,v) &= a\sup_{n\in\mathbb{N}} (1+u_n)^3 + a\sup_{n\in\mathbb{N}} |v_n|^3 + b\sup_{n\in\mathbb{N}} |u_n - v_n|^3 \\ &\geq \left(\frac{4}{3}\right)^3 a = \frac{64}{27}a. \end{aligned}$$

If $a \geq \frac{27}{64}$ *we have* $\|Tu - Tv\|^3 \leq E_3(u,v)$.

Therefore, T satisfies Inequality (4) with $a = \frac{28}{64} = \frac{7}{16}$, $b = \frac{1}{8} \leq \frac{\left(\sqrt[3]{15}-1\right)^3 - 1}{7} \simeq 0.3074$, $p = 3$.
However, T does not satisfy Inequality (4) if $u = \frac{\mathbf{1}}{3}$, $v = \mathbf{0}$ *and* $p = 2$: $\|Tu - Tv\| = 1$ *and*

$$E_2(u,v) = \frac{16}{9}a + a\cdot 0 + \frac{1}{9}b = \frac{16a+b}{9} = \frac{1+14a}{9} < 1.$$

3. Conclusions

In this paper, we generalized the notion of quadratic quasicontraction introduced in Popescu and Stan (Symmetry 2019, 11, 1329 [6]). In the context of the new notion of $(a-p)$-quasicontraction we proved two generalizations of some classical fixed point theorems of Edelstein (J. London Math. Soc. 1962, 37, 74–79 [3]) and Greguš (Boll. Un. Mat. Ital. 1980, 17, 193–198 [5]). Furthermore, we present some examples to support our results.

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