



Relational generalized iterated function systems

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ABSTRACT

In this paper, we introduce a wider class of generalized iterated function systems, called relational generalized iterated function systems. More precisely, the classical contraction condition for functions defined on product spaces is weakened by means of an equivalence relation. In particular, if we consider the total equivalence relation, we recover the classical generalized iterated function systems.

Our main result states that each compact subset of the underlying metric space generates, via a sequence of iterates, a fixed point of the associated fractal operator, called an attractor of the system. We also establish a structure result for the attractors and a theorem concerning the continuous dependence of the attractor on the associated compact set.

Ultimately, we provide some examples which illustrate our main results.

1. Introduction

The concept of iterated function systems (IFSs), which was first introduced by Hutchinson [1] in 1981 and became known through Barnsley [2], holds an important role in the study and analysis of fractal sets. We call an iterated function system on a metric space (X, d) , a finite family of continuous functions $f_1, \dots, f_n : X \rightarrow X$, $n \in \mathbb{N}$, which are usually contractions. Such a system is denoted as $S = ((X, d), \{f_i\}_{i \in \{1, \dots, n\}})$. The classical Hutchinson–Barnsley theorem states that when the underlying metric space is complete, the fractal operator $F_S : P_{cp}(X) \rightarrow P_{cp}(X)$ associated with S , defined as

$$F_S(K) = \bigcup_{i=1}^n f_i(K),$$

for every $K \in P_{cp}(X)$, where $P_{cp}(X)$ denotes the family of all nonempty and compact subsets of X , is a Picard operator, hence it generates a unique fixed point called an attractor or a Hutchinson–Barnsley fractal.

Iterated function systems play an important role in fractal image compression, but they also have various applications in domains such as engineering sciences, medicine, forestry, economy, human anatomy and physics.

The concept under discussion has been generalized in several directions. In this paper, we are concerned with generalized iterated function systems (GIFSs) of order $m \in \mathbb{N}$, i.e. we consider a finite family of maps that are defined on a Cartesian product of m spaces, rather than self-maps. Such systems were first studied by Miculescu and Mihail in [3,4]. See also [5,6]. A significant tool within this generalization is

represented by the generalized code space, which has been developed in [7].

It is worth pointing out that GIFSs constitute a genuine extension of the classical IFSs. In [8], it was shown that for any natural number $m \geq 2$, there exists a subset of the plane which can be represented as the attractor of some GIFS of order m , but is not the attractor of any GIFS of order $m - 1$. See also [9].

In addition to the research regarding the attractors of GIFSs, there have also been studies relating to the Hutchinson measure associated with such systems. In [10], the Hutchinson measure was studied for GIFSs of order two consisting of a countable number of mappings, in [11] its existence was proven for GIFS of order two with place dependent probabilities, and, in [12], its existence was ascertained for GIFS of order $m \in \mathbb{N}$.

Another direction of research involved considering fuzzy GIFSs (see [13]) and GIFSs consisting of F-contractions, rather than usual Banach contractions (see [14]).

These generalized systems were also studied in the case when the framework is different. In [15], GIFSs were considered on Banach spaces, in [16] we find GIFSs in the more general framework of topological spaces and, in [17], GIFSs were studied when the underlying space is a b-metric space.

Using α -dense curves, in [18], the attractor of a GIFS of order $m \in \mathbb{N}$ was approximated using a sequence of finite sets. We also mention that there are a number of papers [19–21] that deal with the representations of the images of attractors of GIFSs.

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In this paper, we introduce a wider class of generalized iterated function systems, called relational generalized iterated function systems (RGIFSs). We replace the classical contraction condition for functions defined on product spaces by a weaker one, which involves an equivalence relation. If we particularize the equivalence relation, namely if we consider the total equivalence relation, we recover the classical generalized iterated function systems.

A RGIFS involves a complete metric space (X, d) , a family of continuous mappings $f_i : X^2 \rightarrow X, i \in I$, where I is a finite set of indices, and a closed equivalence relation R defined on X . We define the fractal operator $F_{S,R} : P_{cp}(X)^2 \rightarrow P_{cp}(X)$ associated with the RGIFS $S = ((X, d), \{f_i\}_{i \in I}, R)$ as

$$F_{S,R}(K_1, K_2) = \bigcup_{i \in I, x \in X} f_i((K_1 \cap [x]) \times (K_2 \cap [x]))$$

for any $K_1, K_2 \in P_{cp}(X)$ such that $\{[x] : x \in K_1\} = \{[x] : x \in K_2\}$, i.e. K_1 and K_2 are equivalent. Via this operator, we define the set function $G_S : P_{cp}(X) \rightarrow P_{cp}(X)$ by

$$G_S(K) = F_{S,R}(K, K),$$

for any $K \in P_{cp}(X)$. The main result of this paper states that G_S is weakly Picard (see Theorems 4 and 7). So, each compact subset $K \in P_{cp}(X)$ generates, by means of the sequence of iterates $(G_S^n(K))_{n \in \mathbb{N}}$, a fixed point of the associated fractal operator $A_K \in P_{cp}(X)$, called an attractor of the system. We also prove a structure result concerning the attractors, i.e. $A_K = \bigcup_{x \in K} A_{[x]}$ (see Theorem 5), and, moreover, we present a result related to the continuous dependence of the attractors on the associated compact set (see Theorem 6).

In the second part of our paper, we are concerned with iterations of a more general kind, i.e. we consider two equivalent compact sets $K_1, K_2 \in P_{cp}(X)$ and define the sequence of iterations $(K_n)_{n \in \mathbb{N}}$ by

$$K_{n+2} = F_{S,R}(K_{n+1}, K_n)$$

for every $n \in \mathbb{N}$. We prove that $(K_n)_{n \in \mathbb{N}}$ converges to the attractor associated with K_1 . (see Theorem 9).

At last, we provide some examples which illustrate our main results. The examples highlight the fact that RGIFSs constitute an extension of GIFSs and we also show that there exist RGIFSs which are not GIFSs.

We emphasize that, regardless of the fact that we consider mappings defined only on X^2 , our results remain valid for the more general case of mappings defined on $X^m, m \in \mathbb{N}$.

2. Preliminaries

By \mathbb{N} we mean the set $\{1, 2, \dots\}$.

If X is any set, $f : X \rightarrow X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we mean the composition of f by itself n times. By $f^{[0]}$ we mean the identity function $Id_X : X \rightarrow X, Id(x) = x$ for all $x \in X$.

Definition 1. Let (X, d) and (Y, ρ) be two metric spaces. The mapping $f : X \rightarrow Y$ is called a contraction if there exists $c \in [0, 1)$ such that

$$\rho(f(x), f(y)) \leq c \cdot d(x, y),$$

for all $x, y \in X$.

The Lipschitz constant of f is

$$Lip(f) = \inf \{c > 0 : \rho(f(x), f(y)) \leq cd(x, y) \text{ for all } x, y \in X\}.$$

Definition 2. Let (X, d) be a metric space. A selfmap $f : X \rightarrow X$ is called:

- (i) a Picard operator if f has a unique fixed point, i.e. there exists a unique element $x_0 \in X$ such that $f(x_0) = x_0$, and, for any $x \in X$, the sequence of iterates $(f^{[n]}(x))_{n \in \mathbb{N}}$ converges to x_0 .
- (ii) a weakly Picard operator if for any $x \in X$, the sequence $(f^{[n]}(x))_{n \in \mathbb{N}}$ converges to a fixed point of f .

Theorem 1 (Banach's Fixed Point Theorem). If (X, d) is a complete metric space and $f : X \rightarrow X$ is a contraction, then f is a Picard operator.

For a metric space (X, d) by:

- $P_{b,cl}(X)$ we mean the family of all nonempty, bounded and closed subsets of X .
- $P_{cp}(X)$ we mean the family of all nonempty and compact subsets of X .

2.1. The Hausdorff–Pompeiu metric

Definition 3. Let (X, d) be a metric space. The function $h_d : P_{b,cl}(X) \times P_{b,cl}(X) \rightarrow [0, \infty)$, or simply h when no confusion should arise, defined by

$$h_d(A, B) = \max \{D(A, B), D(B, A)\},$$

where $D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$ for all $A, B \in P_{b,cl}(X)$, is a metric, called the Hausdorff–Pompeiu metric on $P_{b,cl}(X)$.

Theorem 2 ([2]). If (X, d) is complete, then $(P_{b,cl}(X), h)$ and $(P_{cp}(X), h)$ are likewise complete.

Remark 1. Let (X, d) be a metric space and $A, B \in P_{b,cl}(X)$. If $A \subseteq B$, then $D(A, B) = 0$.

Proposition 1 ([22]). Let (X, d) be a metric space and $(A_i)_{i \in I}, (B_i)_{i \in I} \subseteq P_{b,cl}(X)$ such that $\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i \in P_{b,cl}(X)$. Then

$$h\left(\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i\right) \leq \sup_{i \in I} h(A_i, B_i).$$

Remark 2. Let (X, d) be a metric space, $A, B \in P_{b,cl}(X)$ and $a \in A$. Then, for any $\varepsilon > 0$, there exists $b_\varepsilon \in B$ such that

$$d(a, b_\varepsilon) < h(A, B) + \varepsilon.$$

Proposition 2 ([23]). Let (X, d) be a metric space, $(A_n)_{n \in \mathbb{N}} \subseteq P_{cp}(X)$ and $A \in P_{cp}(X)$ such that $\lim_{n \rightarrow \infty} h(A_n, A) = 0$. Then

$$\bigcup_{n \in \mathbb{N}} A_n \cup A \in P_{cp}(X).$$

2.2. Iterated function systems

Definition 4. A pair $S = ((X, d), (f_i)_{i \in I})$, where (X, d) is a complete metric space, I is a finite set and $f_i : X \rightarrow X, i \in I$, is a finite family of contractions, is called an iterated function system on X (for short IFS).

Such a system induces a set function $F_S : P_{cp}(X) \rightarrow P_{cp}(X)$, called the fractal or the Hutchinson operator associated with S , given by

$$F_S(K) = \bigcup_{i \in I} f_i(K),$$

for all $K \in P_{cp}(X)$.

The set function F_S turns out to be a contraction on the complete metric space $(P_{cp}(X), h)$, therefore, according to Banach's Fixed Point Theorem, there exists a unique set $A_S \in P_{cp}(X)$ such that

$$A_S = F_S(A_S) = \bigcup_{i \in I} f_i(A_S),$$

and, for any $K \in P_{cp}(X)$, the sequence of iterates $(F_S^{[n]}(K))_{n \in \mathbb{N}}$ converges to A_S , i.e.

$$\lim_{n \rightarrow \infty} h(F_S^{[n]}(K), A_S) = 0.$$

The set A_S is called the attractor of S .

2.3. Generalized iterated function systems

Let (X, d) be a metric space. Then the Cartesian product space $X^2 = X \times X$ is a metric space under the maximum metric:

$$d_{max}((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\},$$

for all $x_1, y_1, x_2, y_2 \in X$.

If (X, d) is complete, then (X^2, d_{max}) is likewise complete.

The following theorem is the corresponding of Banach's Fixed Point Principle for mappings defined on a Cartesian product of metric spaces.

Theorem 3 ([24]). *Let (X, d) be a complete metric space and $f : X^2 \rightarrow X$ a contraction. Then there exists a unique $x^* \in X$ such that*

$$f(x^*, x^*) = x^*$$

and, for every $x_1, x_2 \in X$, the sequence $(x_n)_{n \in \mathbb{N}}$, defined by $x_{n+2} = f(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, has the property that

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Concerning the speed of convergence, we have the following estimation:

$$d(x_n, x^*) \leq 2 \frac{c^{\lfloor \frac{n}{2} \rfloor}}{1 - c} \max\{d(x_1, x_2), d(x_2, x_3)\},$$

for every $n \in \mathbb{N}$, where $c = Lip(f)$.

Definition 5. A pair $S = ((X^2, d), \{f_i\}_{i \in I})$, where (X, d) is a complete metric space, I is a finite set and $f_i : X \times X \rightarrow X$, $i \in I$, is a finite family of contractions, is called a generalized iterated function system (for short GIFS) of order 2 on X .

Such a system induces a set function $F_S : P_{cp}(X) \times P_{cp}(X) \rightarrow P_{cp}(X)$, called the fractal or the Hutchinson operator associated with S , given by

$$F_S(K_1, K_2) = \bigcup_{i \in I} f_i(K_1 \times K_2),$$

for all $K_1, K_2 \in P_{cp}(X)$.

The set function F_S turns out to be a contraction on the complete metric space $P_{cp}(X) \times P_{cp}(X)$, therefore, according to [Theorem 3](#), there exists a unique set $A_S \in P_{cp}(X)$ such that

$$A_S = F_S(A_S, A_S) = \bigcup_{i \in I} f_i(A_S \times A_S),$$

and, for any $K_1, K_2 \in P_{cp}(X)$, the sequence $(K_n)_{n \in \mathbb{N}}$, described by $K_{n+2} = F_S(K_{n+1}, K_n)$ for every $n \in \mathbb{N}$, converges to A_S , i.e.

$$\lim_{n \rightarrow \infty} h(K_n, A_S) = 0.$$

We have the following estimation for the speed of convergence:

$$h(K_n, A_S) \leq 2 \frac{c^{\lfloor \frac{n}{2} \rfloor}}{1 - c} \cdot \max\{h(K_1, K_2), h(K_2, K_3)\},$$

for all $n \in \mathbb{N}$, where $c = \max_{i \in I} Lip(f_i)$.

The set A_S is called the attractor of S .

We refer the reader to [4] for more details.

2.4. The code space for GIFS of order 2

In this subsection we present the generalized code space related to GIFSs of order 2. Our notation is based on that given in [7].

Let I be a nonempty finite set and let us define inductively the following sequence of sets:

- $I_1 = I$
- $I_k = I_{k-1} \times I_{k-1}$ for $k \in \mathbb{N}$, $k \geq 2$.

For $k \in \mathbb{N}$, the family of k -words (words of length k) is

$${}_k I = I_1 \times I_2 \times \dots \times I_k.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in {}_k I$ and $m \leq k$, $m \in \mathbb{N}$, then we denote $\alpha|_m = (\alpha_1, \alpha_2, \dots, \alpha_m)$.

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) = (\alpha_1, (\alpha_2^1, \alpha_2^2), \dots, (\alpha_k^1, \alpha_k^2)) \in {}_k I$, where $k \in \mathbb{N}$, $k \geq 2$, the two left shifts of α are defined as

$$\alpha(i) = (\alpha_2^i, \alpha_3^i, \dots, \alpha_k^i) \in {}_{k-1} I,$$

where $i \in \{1, 2\}$.

For a metric space (or any set) X , define the following sequence of sets:

- $X_1 = X^2$
- $X_k = X_{k-1} \times X_{k-1}$ for $k \in \mathbb{N}$, $k \geq 2$.

Starting from the finite family $f_i : X^2 \rightarrow X$, $i \in I$, we define inductively the generalized compositions of functions from this family $f_\alpha : X_k \rightarrow X$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in {}_k I$, where $k \in \mathbb{N}$, as

$$f_\alpha(x_1, x_2) = f_{\alpha_1}(f_{\alpha(1)}(x_1), f_{\alpha(2)}(x_2)),$$

for all $(x_1, x_2) \in X_k$.

Let us now define inductively a natural isometry $h_n : X^{2^n} \rightarrow X_n$, $n \in \mathbb{N}$, that places elements of X^{2^n} , i.e. elements that have 2^n components, into brackets such that they belong to X_n .

First, for $x = (x_1, x_2, \dots, x_{2^{n+1}}) \in X^{2^{n+1}}$ and $i \in \{1, 2\}$, denote by $x(i) = (x_{(i-1)2^n+1}, \dots, x_{i2^n})$ the i th block of x . Note that $x(i) \in X^{2^n}$.

Now set

$$h_1(x_1, x_2) = (x_1, x_2)$$

for $(x_1, x_2) \in X^2$ and

$$h_n(x) = (h_{n-1}(x(1)), h_{n-1}(x(2)))$$

for $x \in X^{2^n}$ and $n \in \mathbb{N}$, $n \geq 2$.

3. Relational generalized iterated function systems

Definition 6. A triplet $S = ((X, d), \{f_i\}_{i \in I}, R)$, where (X, d) is a complete metric space, I is an arbitrarily chosen finite set, R is a closed equivalence relation on X and $f_i : X^2 \rightarrow X$ is continuous for any $i \in I$, is called a relational generalized iterated function system (for short RGIFS) if:

- (i) there exists $c \in [0, 1)$ such that

$$d(f_i(x_1, y_1), f_i(x_2, y_2)) \leq c \cdot \max\{d(x_1, x_2), d(y_1, y_2)\},$$

for any $x_1, y_1, x_2, y_2 \in X$ such that $x_1 R x_2$, $y_1 R y_2$ and $i \in I$.

- (ii) for any $x_1, x_2 \in X$ and $i \in I$, if $x_1 R x_2$, then $f_i(x_1, x_2) R x_1$.

Remark 3. On the one hand, if the constitutive functions of a RGIFS are contractions, then condition (i) from [Definition 6](#) is obviously fulfilled. On the other hand, as we can see in the second example from [Section 5](#), the constitutive functions of a RGIFS need not be Lipschitz. Moreover, although they may be Lipschitz, their Lipschitz constant can be 1.

Remark 4. In the framework of [Definition 6](#), if $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are two sequences of elements from X such that

- (i) $x_n R y_n$ for every $n \in \mathbb{N}$;
- (ii) there exist $x, y \in X$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$,

then $x R y$.

Let $S = ((X, d), \{f_i\}_{i \in I}, R)$ be a RGIFS. We shall use the following notation:

- Sometimes, we denote xRy by $x \sim y$, where $x, y \in X$.
- $[x] = \{y \in X : xRy\}$, where $x \in X$.
- $[K] = \{[x] : x \in K\}$, where $K \in P_{cp}(X)$.
- $K^x = K \cap [x]$, where $x \in X$ and $K \in P_{cp}(X)$.
- $K_1 \overset{R}{\sim} K_2$ if $[K_1] = [K_2]$, where $K_1, K_2 \in P_{cp}(X)$.
- $\mathcal{M}_R = \{(K_1, K_2) \in P_{cp}(X) \times P_{cp}(X) : K_1 \overset{R}{\sim} K_2\}$.
- $F_{S,R}(K_1, K_2) = \bigcup_{i \in I, x \in X} f_i(K_1^x \times K_2^x) = \bigcup_{i \in I, x \in K_1} f_i(K_1^x \times K_2^x) = \bigcup_{i \in I, x \in K_2} f_i(K_1^x \times K_2^x)$, for any $(K_1, K_2) \in \mathcal{M}_R$.
- $\tilde{K} = \{H \in P_{cp}(X) : H \overset{R}{\sim} K\}$, where $K \in P_{cp}(X)$.
- $C_K = \text{diam}(K \cup \bigcup_{i \in I} f_i(K \times K))$ for any $K \in P_{cp}(X)$.

Lemma 1. For $K_1, K_2 \in P_{cp}(X)$, the following statements are equivalent:

- (i) $K_1 \overset{R}{\sim} K_2$.
- (ii) For any $x \in X$ the following equivalence is valid:
 $K_1 \cap [x] \neq \emptyset \Leftrightarrow K_2 \cap [x] \neq \emptyset$.

Proof. (i) \Rightarrow (ii) Let $K_1, K_2 \in P_{cp}(X)$ be such that $K_1 \overset{R}{\sim} K_2$, i.e. $[K_1] = [K_2]$. If $x \in X$ is such that $K_1 \cap [x] \neq \emptyset$, then there exists $y \in K_1$ such that $[x] = [y]$. Hence $[y] \in [K_1] = [K_2]$, so there exists $z \in K_2$ such that $z \in [y] = [x]$, therefore $K_2 \cap [x] \neq \emptyset$. The reverse implication can be proved in a similar manner. (ii) \Rightarrow (i) For $y \in K_1$ we have $K_1 \cap [y] \neq \emptyset$ and the hypothesis implies $K_2 \cap [y] \neq \emptyset$, so there exists $z \in K_2$ such that $z \in [y]$. Hence $[z] = [y] \in [K_2]$. Consequently, we proved that $[K_1] \subseteq [K_2]$. The reverse inclusion can be proved in a similar way. \square

Remark 5. Note that

$$f_i([x] \times [x]) \subseteq [x],$$

for any $x \in X$ and $i \in I$.

Proposition 3. $F_{S,R}(K_1, K_2) \in P_{cp}(X)$ for any $(K_1, K_2) \in \mathcal{M}_R$.

Proof. Claim. $\bigcup_{x \in K_1} f_i(K_1^x \times K_2^x)$ is compact for any $i \in I$ and $(K_1, K_2) \in \mathcal{M}_R$.

Justification of Claim: Fix $i \in I, (K_1, K_2) \in \mathcal{M}_R$ and let $(x_n)_{n \in \mathbb{N}} \subseteq \bigcup_{x \in K_1} f_i(K_1^x \times K_2^x)$. Then, for any $n \in \mathbb{N}$, there exist $y_n \in K_1, u_n^1 \in K_1 \cap [y_n], u_n^2 \in K_2 \cap [y_n]$ such that

$$x_n = f_i(u_n^1, u_n^2) \in f_i([y_n] \times [y_n]) \stackrel{\text{Remark 5}}{\subseteq} [y_n].$$

Since $K_1, K_2 \in P_{cp}(X)$, we deduce that

- (i) there exists $(u_{n_k}^1)_{k \in \mathbb{N}} \subseteq (u_n^1)_{n \in \mathbb{N}}$ and $u_1 \in K_1$ such that $\lim_{k \rightarrow \infty} u_{n_k}^1 = u_1$;
- (ii) there exists $(u_{n_k}^2)_{k \in \mathbb{N}} \subseteq (u_n^2)_{n \in \mathbb{N}}$ and $u_2 \in K_2$ such that $\lim_{k \rightarrow \infty} u_{n_k}^2 = u_2$;
- (iii) there exists $(y_{n_k}^q)_{q \in \mathbb{N}} \subseteq (y_{n_k}^p)_{p \in \mathbb{N}}$ and $y \in K_1$ such that $\lim_{q \rightarrow \infty} y_{n_k}^q = y$.

Since

$$\lim_{q \rightarrow \infty} u_{n_k}^1 = u_1, \lim_{q \rightarrow \infty} y_{n_k}^q = y, u_{n_k}^1 \in [y_{n_k}^q], \text{ for every } q \in \mathbb{N},$$

via Remark 4, we deduce that $u_1 \in K_1 \cap [y]$.

Similarly we have that $u_2 \in K_2 \cap [y]$.

Finally, since f_i is continuous, we see that

$$\lim_{q \rightarrow \infty} x_{n_k}^q = \lim_{q \rightarrow \infty} f_i(u_{n_k}^1, u_{n_k}^2) = f_i(u_1, u_2) \in \bigcup_{x \in K_1} f_i(K_1^x \times K_2^x).$$

So $\bigcup_{x \in K_1} f_i(K_1^x \times K_2^x)$ is sequentially compact, hence compact, for each $(K_1, K_2) \in \mathcal{M}_R$ and $i \in I$, and the justification of the Claim is done.

Consequently,

$$F_{S,R}(K_1, K_2) = \bigcup_{i \in I, x \in K_1} f_i(K_1^x \times K_2^x) \stackrel{\text{Claim \& I finite}}{\in} P_{cp}(X),$$

for any $(K_1, K_2) \in \mathcal{M}_R$. \square

Therefore, we can define the function $F_{S,R} : \mathcal{M}_R \rightarrow P_{cp}(X)$ as follows:

$$F_{S,R}(K_1, K_2) = \bigcup_{i \in I, x \in X} f_i(K_1^x \times K_2^x),$$

for any $(K_1, K_2) \in \mathcal{M}_R$.

We can also define the function $G_S : P_{cp}(X) \rightarrow P_{cp}(X)$, given by

$$G_S(K) = F_{S,R}(K, K),$$

for any $K \in P_{cp}(X)$.

For any $K \in P_{cp}(X)$ we consider the sequence $(G_S^{[n]}(K))_{n \in \mathbb{N} \cup \{0\}}$, where $G_S^{[0]}(K) = K$.

Remark 6. We have

$$\begin{aligned} G_S(K) &= \bigcup_{x_1, x_2 \in K} F_{S,R}(\{x_1\}, \{x_2\}) = \\ &= \bigcup_{x_1, x_2 \in K, i \in I, y \in X} f_i(\{x_1\} \cap [y] \times \{x_2\} \cap [y]) \\ &= \bigcup_{x_1, x_2 \in K, x_1 \sim x_2, i \in I} \{f_i(x_1, x_2)\}, \end{aligned}$$

for all $K \in P_{cp}(X)$.

4. Main results

Throughout this section let $S = ((X, d), \{f_i\}_{i \in I}, R)$ be a RGIFS.

Theorem 4. For any $K \in P_{cp}(X)$ there exists $A_K \in P_{cp}(X)$ such that

$$\lim_{n \rightarrow \infty} G_S^{[n]}(K) = A_K$$

and

$$h(G_S^{[n]}(K), A_K) \leq \frac{c^n}{1-c} \cdot C_K,$$

for all $n \in \mathbb{N}$.

Proof. Let $K \in P_{cp}(X)$ be fixed, but arbitrarily chosen.

Claim1.

$$G_S^{[n]}(K) = \bigcup_{y \in K, x \in [y]_n \cap K_n, \alpha \in_n I} \{f_\alpha(x)\},$$

for all $n \in \mathbb{N}$.

Justification of Claim 1: The above assertion is true for $n = 1$ since

$$\begin{aligned} \bigcup_{y \in K, x \in [y]_1 \cap K_1, i \in I} \{f_i(x)\} &= \bigcup_{y \in K, x_1, x_2 \in K^y, i \in I} \{f_i(x_1, x_2)\} = \\ &= \bigcup_{x_1, x_2 \in K, x_1 \sim x_2, i \in I} \{f_i(x_1, x_2)\} \stackrel{\text{Remark 6}}{=} G_S(K). \end{aligned} \tag{1}$$

Suppose now that the assertion is true for $n \in \mathbb{N}$.

On the one hand, we have

$$G_S^{[n+1]}(K) \subseteq \bigcup_{y \in K, x \in [y]_{n+1} \cap K_{n+1}, \alpha \in_{n+1} I} \{f_\alpha(x)\}. \tag{2}$$

Indeed, observe first that

$$G_S^{[n+1]}(K) = G_S(G_S^{[n]}(K)) \stackrel{(1)}{=} \bigcup_{y_1, y_2 \in G_S^{[n]}(K), y_1 \sim y_2, i \in I} \{f_i(y_1, y_2)\}.$$

If $y_1, y_2 \in G_S^{[n]}(K)$ are such that $y_1 \sim y_2$, then there exist $\alpha, \beta \in {}_n I$, $u_1, u_2 \in K$, $x_1 \in [u_1]_n \cap K_n$ and $x_2 \in [u_2]_n \cap K_n$ such that $y_1 = f_{(\alpha_1, \dots, \alpha_n)}(x_1)$ and $y_2 = f_{(\beta_1, \dots, \beta_n)}(x_2)$. Since $y_1 \sim y_2$, via Definition 6, (ii), we deduce that $[u_1] = [u_2]$. For all $i \in I$, we have

$$f_i(y_1, y_2) = f_i \left(f_{(\alpha_1, \dots, \alpha_n)}(x_1), f_{(\beta_1, \dots, \beta_n)}(x_2) \right) = f_{(i, (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))}(x_1, x_2) = f_\omega(x),$$

where $\omega = (i, (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)) \in {}_{n+1} I$ and $x = (x_1, x_2) \in [u_1]_{n+1} \cap K_{n+1}$, so

$$f_\omega(x) \in \bigcup_{y \in K, x \in [y]_{n+1} \cap K_{n+1}, \alpha \in {}_{n+1} I} \{f_\alpha(x)\},$$

and the justification of (2) is done.

On the other hand, we have

$$\bigcup_{y \in K, x \in [y]_{n+1} \cap K_{n+1}, \alpha \in {}_{n+1} I} \{f_\alpha(x)\} \subseteq G_S^{[n+1]}(K). \tag{3}$$

Indeed, if $u \in \bigcup_{y \in K, x \in [y]_{n+1} \cap K_{n+1}, \alpha \in {}_{n+1} I} \{f_\alpha(x)\}$, then there exist $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in {}_{n+1} I$, $y \in K$ and $x = (x_1, x_2) \in [y]_{n+1} \cap K_{n+1}$ such that

$$u = f_\alpha(x) = f_{\alpha(1)}(x_1), f_{\alpha(2)}(x_2).$$

Taking into account that $x_1, x_2 \in [y]_n \cap K_n$, $\alpha(1), \alpha(2) \in {}_n I$, $f_{\alpha(1)}(x_1), f_{\alpha(2)}(x_2) \in G_S^{[n]}(K)$ and $f_{\alpha(1)}(x_1) \sim f_{\alpha(2)}(x_2)$, we infer that

$$u \in \bigcup_{y_1, y_2 \in G_S^{[n]}(K), y_1 \sim y_2, i \in I} \{f_i(y_1, y_2)\} = G_S(G_S^{[n]}(K)) = G_S^{[n+1]}(K),$$

and the justification of (3) is complete.

In view of (2) and (3), we conclude that the assertion is true for $n + 1$, so the justification of Claim 1 is done.

Claim2. For any $K \in P_{cp}(X)$, $y \in X$, $n \in \mathbb{N}$, $x, y_1, y_2 \in [y]_n \cap K_n$, $(\alpha_1, \dots, \alpha_n) \in {}_n I$ and $\gamma \in I_{n+1}$ we have

$$d \left(f_{(\alpha_1, \dots, \alpha_n)}(x), f_{(\alpha_1, \dots, \alpha_n, \gamma)}(y_1, y_2) \right) \leq c^n \cdot C_K.$$

Justification of Claim 2: The assertion is true for $n = 1$ since

$$d \left(f_i(x_1, x_2), f_i \left(f_{j_1}(y_1), f_{j_2}(y_2) \right) \right) \leq c \cdot \max \left\{ d(x_1, f_{j_1}(y_1)), d(x_2, f_{j_2}(y_2)) \right\} \leq c \cdot C_K,$$

for all $i \in I$, $\gamma = (j_1, j_2) \in I_2$, $y \in X$ and $x = (x_1, x_2), y_1, y_2 \in [y]_1 \cap K_1$.

Suppose now that the assertion is true for $n \in \mathbb{N}$. Then

$$\begin{aligned} & d \left(f_{(\alpha_1, \dots, \alpha_n, \alpha_{n+1})}(x_1, x_2), f_{(\alpha_1, \dots, \alpha_n, \gamma)}(y_1, y_2) \right) = \\ & = d \left(f_{\alpha_1} \left(f_{(\alpha_2^1, \dots, \alpha_{n+1}^1)}(x_1), f_{(\alpha_2^2, \dots, \alpha_{n+1}^2)}(x_2) \right), \right. \\ & \quad \left. f_{\alpha_1} \left(f_{(\alpha_2^1, \dots, \alpha_{n+1}^1, \gamma^1)}(y_1), f_{(\alpha_2^2, \dots, \alpha_{n+1}^2, \gamma^2)}(y_2) \right) \right) \leq \\ & \leq c \cdot \max \left\{ d \left(f_{(\alpha_2^1, \dots, \alpha_{n+1}^1)}(x_1), f_{(\alpha_2^1, \dots, \alpha_{n+1}^1, \gamma^1)}(y_1) \right), \right. \\ & \quad \left. d \left(f_{(\alpha_2^2, \dots, \alpha_{n+1}^2)}(x_2), f_{(\alpha_2^2, \dots, \alpha_{n+1}^2, \gamma^2)}(y_2) \right) \right\} \leq \\ & \leq c^{n+1} \cdot C_K, \end{aligned}$$

for all $y \in X$, $x = (x_1, x_2), y_1, y_2 \in [y]_{n+1} \cap K_{n+1}$, $(\alpha_1, \dots, \alpha_{n+1}) \in {}_{n+1} I$ and $\gamma \in I_{n+2}$, so the justification of Claim 2 is complete.

Claim3. For any $K \in P_{cp}(X)$ and $n \in \mathbb{N}$, we have

$$h \left(G_S^{[n]}(K), G_S^{[n+1]}(K) \right) \leq c^n \cdot C_K.$$

Justification of Claim 3: On the one hand, we have

$$D \left(G_S^{[n]}(K), G_S^{[n+1]}(K) \right) \leq c^n \cdot C_K. \tag{4}$$

Indeed, for $z_1 \in G_S^{[n]}(K)$, according to Claim 1, there exist $y \in K$, $x = (x_1, x_2) \in [y]_n \cap K_n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in {}_n I$ such that

$$z_1 = f_\alpha(x_1, x_2).$$

Setting

$$z_2 = f_{(\alpha_1, \dots, \alpha_n, (\alpha_n, \alpha_n))} \left((x_1, x_2), (x_1, x_2) \right) \stackrel{\text{Claim 1}}{\in} G_S^{[n+1]}(K),$$

we get

$$d \left(z_1, G_S^{[n+1]}(K) \right) \leq d(z_1, z_2) \stackrel{\text{Claim 2}}{\leq} c^n \cdot C_K.$$

On the other hand, we have

$$D \left(G_S^{[n+1]}(K), G_S^{[n]}(K) \right) \leq c^n \cdot C_K. \tag{5}$$

Indeed for $z_2 \in G_S^{[n+1]}(K)$, taking into account Claim 1, there exist $y \in K$ and $(u_1, u_2), (v_1, v_2) \in [y]_n \cap K_n$ such that

$$z_2 = f_{(\alpha_1, \alpha_2, \dots, \alpha_{n+1})} \left((u_1, u_2), (v_1, v_2) \right).$$

Then

$$z_1 = f_{(\alpha_1, \dots, \alpha_n)}(u_1, u_2) \stackrel{\text{Claim 1}}{\in} G_S^{[n]}(K)$$

and

$$d(z_1, z_2) \stackrel{\text{Claim 2}}{\leq} c^n \cdot C_K.$$

In view of (4) and (5), the justification of Claim 3 is complete.

Now, for any $n, p \in \mathbb{N}$ and $K \in P_{cp}(X)$, we deduce that

$$\begin{aligned} h \left(G_S^{[n]}(K), G_S^{[n+p]}(K) \right) & \leq \sum_{k=n}^{n+p-1} h \left(G_S^{[k]}(K), G_S^{[k+1]}(K) \right) \stackrel{\text{Claim 3}}{\leq} \sum_{k=n}^{n+p-1} c^k \cdot C_K = \\ & = c^n \frac{1-c^p}{1-c} \cdot C_K < \frac{c^n}{1-c} \cdot C_K. \end{aligned} \tag{6}$$

As $\lim_{n \rightarrow \infty} \frac{c^n}{1-c} \cdot C_K = 0$, we infer that $\left(G_S^{[n]}(K) \right)_{n \in \mathbb{N}}$ is a Cauchy sequence, so there exists $A_K \in P_{cp}(X)$ such that

$$\lim_{n \rightarrow \infty} G_S^{[n]}(K) = A_K.$$

Moreover, note that, in view of 4, we obtain that

$$h \left(G_S^{[n]}(K), A_K \right) \leq \frac{c^n}{1-c} \cdot C_K$$

for any $K \in P_{cp}(X)$ and $n \in \mathbb{N}$. \square

Lemma 2. Suppose that $(A_n)_{n \in \mathbb{N}} \subseteq P_{cp}(X)$ and $A \in P_{cp}(X)$ are such that $\lim_{n \rightarrow \infty} h(A_n, A) = 0$. If $B \in P_{cl}(X)$ has the property that $A_n \subseteq B$ for every $n \in \mathbb{N}$, then $A \subseteq B$.

Proof. As $\lim_{n \rightarrow \infty} h(A_n, A) = 0$, we have

$$\lim_{n \rightarrow \infty} d(x, A_n) = 0,$$

for every $x \in A$.

Note that

$$d(x, B) \stackrel{A_n \subseteq B}{\leq} d(x, A_n),$$

for every $n \in \mathbb{N}$ and $x \in X$.

Hence, we deduce that $d(x, B) = 0$, i.e. $x \in B$ for every $x \in A$, therefore $A \subseteq B$. \square

Lemma 3. For any $x \in X$, we have

$$A_{\{x\}} \subseteq [x].$$

Proof. Fix $n \in \mathbb{N}$ and observe that, via Claim 1 from the proof of Theorem 4, we have

$$G_S^{[n]}(\{x\}) = \bigcup_{y \in \{x\}, u \in [y]_n \cap \{x\}_n, \alpha \in {}_n I} \{f_\alpha(u)\} = \bigcup_{\alpha \in {}_n I} \{f_\alpha(h_n(\overline{x}_n))\}, \tag{7}$$

where $\overline{x}_n = (x, x, \dots, x) \in X^{2^n}$.

Now, since

$$G_S^{[n]}(\{x\}) \subseteq [x],$$

for any $n \in \mathbb{N}$, in view of Lemma 2, we conclude that

$$\lim_{n \rightarrow \infty} G_S^{[n]}(\{x\}) = A_{\{x\}} \subseteq [x]. \quad \square$$

Lemma 4. For any $x, y \in X$, the following statements are equivalent:

- (i) $A_{\{x\}} = A_{\{y\}}$;
- (ii) $[x] = [y]$.

Proof. Let $x, y \in X$ be fixed, but arbitrarily chosen.

(i) \Rightarrow (ii) Since $A_{\{x\}} \subseteq [x]$ and $A_{\{x\}} \subseteq [y]$, we deduce that $[x] \cap [y] \neq \emptyset$, hence $[x] = [y]$. (ii) \Rightarrow (i) On the one hand, we have

$$\begin{aligned} h(A_{\{x\}}, A_{\{y\}}) &\stackrel{(7)}{=} \lim_{n \rightarrow \infty} h\left(\bigcup_{\alpha \in_n I} \{f_\alpha(h_n(\overline{x_n}))\}, \bigcup_{\alpha \in_n I} \{f_\alpha(h_n(\overline{y_n}))\}\right) \stackrel{\text{Proposition 1}}{\leq} \\ &\leq \lim_{n \rightarrow \infty} \sup_{\alpha \in_n I} d(f_\alpha(h_n(\overline{x_n})), f_\alpha(h_n(\overline{y_n}))). \end{aligned}$$

On the other hand, for every $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in_n I$, we have

$$\begin{aligned} d(f_\alpha(h_n(\overline{x_n})), f_\alpha(h_n(\overline{y_n}))) &= \\ &= d\left(f_{\alpha_1}(f_{\alpha(1)}(h_{n-1}(\overline{x_{n-1}})), f_{\alpha(2)}(h_{n-1}(\overline{x_{n-1}}))), \right. \\ &\quad \left. f_{\alpha_1}(f_{\alpha(1)}(h_{n-1}(\overline{y_{n-1}})), f_{\alpha(2)}(h_{n-1}(\overline{y_{n-1}})))\right) \leq \\ &\leq c \cdot \max\left\{d\left(f_{\alpha(1)}(h_{n-1}(\overline{x_{n-1}})), f_{\alpha(1)}(h_{n-1}(\overline{y_{n-1}}))\right), \right. \\ &\quad \left. d\left(f_{\alpha(2)}(h_{n-1}(\overline{x_{n-1}})), f_{\alpha(2)}(h_{n-1}(\overline{y_{n-1}}))\right)\right\} \leq \\ &\leq \dots \leq c^n \cdot d(x, y). \end{aligned}$$

Consequently

$$\lim_{n \rightarrow \infty} \sup_{\alpha \in_n I} d(f_\alpha(h_n(\overline{x_n})), f_\alpha(h_n(\overline{y_n}))) = 0,$$

hence

$$A_{\{x\}} = A_{\{y\}}. \quad \square$$

Lemma 5. The function $t : X \rightarrow P_{cp}(X)$, given by

$$t(x) = A_{\{x\}},$$

for all $x \in X$, is continuous on each $K \in P_{cp}(X)$.

Proof. Claim1. $f_\alpha : X_k \rightarrow X$ is continuous for every $\alpha \in_k I$ and $k \in \mathbb{N}$.

Justification of Claim 1: For $k = 1$ the assertion is trivially true, since $f_i : X^2 \rightarrow X$ is continuous for any $i \in I$.

Suppose that the assertion is valid for $k \in \mathbb{N}$. For $\alpha = (\alpha_1, \dots, \alpha_{k+1}) \in_{k+1} I$, $(x_n = (x_n^1, x_n^2))_{n \in \mathbb{N}} \subseteq X_{k+1}$ and $x = (x_1, x_2) \in X_{k+1}$ such that $\lim_{n \rightarrow \infty} x_n = x$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} f_{(\alpha_1, \dots, \alpha_{k+1})}(x_n) &= \lim_{n \rightarrow \infty} f_{\alpha_1}(f_{\alpha(1)}(x_n^1), f_{\alpha(2)}(x_n^2)) = \\ &= f_{\alpha_1}\left(\lim_{n \rightarrow \infty} f_{\alpha(1)}(x_n^1), \lim_{n \rightarrow \infty} f_{\alpha(2)}(x_n^2)\right) = \\ &= f_{\alpha_1}(f_{\alpha(1)}(x_1), f_{\alpha(2)}(x_2)) = f_{(\alpha_1, \dots, \alpha_{k+1})}(x). \end{aligned}$$

Hence the assertion is also true for $k + 1$ and the justification of Claim 1 is done.

Let us define $t_n : X \rightarrow P_{cp}(X)$, $n \in \mathbb{N}$, as follows

$$t_n(x) = G_S^{[n]}(\{x\}),$$

for all $x \in X$.

Claim2. The sequence $(t_n)_{n \in \mathbb{N}}$ converges uniformly to t on each $K \in P_{cp}(X)$.

Justification of Claim 2: Let $K \in P_{cp}(X)$ and note that

$$h(t_n(x), t(x)) = h\left(G_S^{[n]}(\{x\}), A_{\{x\}}\right) \stackrel{\text{Theorem 4}}{\leq} \frac{c^n}{1-c} \cdot C_{\{x\}} \leq \frac{c^n}{1-c} \cdot C_K$$

for all $x \in K$ and $n \in \mathbb{N}$.

Hence

$$\sup_{x \in K} h(t_n(x), t(x)) \leq \frac{c^n}{1-c} \cdot C_K,$$

for every $n \in \mathbb{N}$.

Consequently

$$\lim_{n \rightarrow \infty} \sup_{x \in K} h(t_n(x), t(x)) = 0$$

and the justification of Claim 2 is complete.

Claim3. t_n is continuous for every $n \in \mathbb{N}$.

Justification of Claim 3: Let $n \in \mathbb{N}$ and $x \in X$ be fixed, but arbitrarily chosen. Note that if $(x_m)_{m \in \mathbb{N}} \subseteq X$ is such that $\lim_{m \rightarrow \infty} x_m = x$, then we have

$$\begin{aligned} h(t_n(x_m), t_n(x)) &= h\left(G_S^{[n]}(\{x_m\}), G_S^{[n]}(\{x\})\right) \stackrel{(7)}{=} \\ &= h\left(\bigcup_{\alpha \in_n I} \{f_\alpha(h_n(\overline{(x_m)_n}))\}, \bigcup_{\alpha \in_n I} \{f_\alpha(h_n(\overline{x_n}))\}\right) \leq \\ &\leq \max_{\alpha \in_n I} d\left(f_\alpha(h_n(\overline{(x_m)_n})), f_\alpha(h_n(\overline{x_n}))\right), \end{aligned}$$

for every $n \in \mathbb{N}$ and, by passing to limit as $m \rightarrow \infty$ in the above relation, using Claim 1, we deduce that

$$\lim_{m \rightarrow \infty} t_n(x_m) = t_n(x).$$

Thus the justification of Claim 3 is complete.

In view of Claim 2 and Claim 3, the proof of Lemma 5 is done. \square

Remark 7. Taking into account Lemma 5, Theorem 2.1 from [25] and the diagram on page 396 of [26], we infer that

$$\bigcup_{x \in K} t(x) = \bigcup_{x \in K} A_{\{x\}} \in P_{cp}(X),$$

for every $K \in P_{cp}(X)$.

Theorem 5. We have

$$A_K = \bigcup_{x \in K} A_{\{x\}},$$

for every $K \in P_{cp}(X)$.

Proof. Let $K \in P_{cp}(X)$ be fixed, but arbitrarily chosen.

Claim.

$$\lim_{n \rightarrow \infty} h\left(G_S^{[n]}(K), \bigcup_{x \in K} G_S^{[n]}(\{x\})\right) = 0.$$

Justification of Claim: Using Claim 1 from the proof of Theorem 4, we deduce that

$$\begin{aligned} h\left(G_S^{[n]}(K), \bigcup_{x \in K} G_S^{[n]}(\{x\})\right) &= h\left(\bigcup_{x \in K, y \in [x]_n \cap K_n, \alpha \in_n I} \{f_\alpha(y)\}, \bigcup_{x \in K} G_S^{[n]}(\{x\})\right) \stackrel{\text{Proposition 1}}{\leq} \\ &\leq \sup_{x \in K} h\left(\bigcup_{y \in [x]_n \cap K_n, \alpha \in_n I} \{f_\alpha(y)\}, G_S^{[n]}(\{x\})\right) \stackrel{(7)}{=} \\ &= \sup_{x \in K} h\left(\bigcup_{y \in [x]_n \cap K_n, \alpha \in_n I} \{f_\alpha(y)\}, \bigcup_{\alpha \in_n I} \{f_\alpha(h_n(\overline{x_n}))\}\right) \stackrel{\text{Proposition 1}}{\leq} \\ &\leq \sup_{x \in K, \alpha \in_n I} h\left(\bigcup_{y \in [x]_n \cap K_n} \{f_\alpha(y)\}, \{f_\alpha(h_n(\overline{x_n}))\}\right) \stackrel{\text{Remark 1}}{=} \\ &= \sup_{x \in K, \alpha \in_n I} D\left(\bigcup_{y \in [x]_n \cap K_n} \{f_\alpha(y)\}, \{f_\alpha(h_n(\overline{x_n}))\}\right), \end{aligned} \tag{8}$$

for all $n \in \mathbb{N}$, where $\overline{x_n} = (x, x, \dots, x) \in X^{2^n}$.

For $u = f_\alpha(y)$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in_n I$, $y = (y_1, y_2) \in [x]_n \cap K_n$ and $x \in K$, we obtain that

$$\begin{aligned} d(u, f_\alpha(h_n(\overline{x_n}))) &= d\left(f_{\alpha_1}(f_{\alpha(1)}(y_1), f_{\alpha(2)}(y_2)), \right. \\ &\quad \left. f_{\alpha_1}(f_{\alpha(1)}(h_{n-1}(\overline{x_{n-1}})), f_{\alpha(2)}(h_{n-1}(\overline{x_{n-1}})))\right) \leq \end{aligned}$$

$$\begin{aligned} &\leq c \cdot \max \left\{ d \left(f_{\alpha(1)}(y_1), f_{\alpha(1)}(h_{n-1}(\overline{x_{n-1}})) \right), \right. \\ &\quad \left. d \left(f_{\alpha(2)}(y_2), f_{\alpha(2)}(h_{n-1}(\overline{x_{n-1}})) \right) \right\} \leq \\ &\leq \dots \leq c^n \cdot \text{diam}(K), \end{aligned}$$

so

$$\sup_{x \in K, a \in I} D \left(\bigcup_{y \in [x]_n \cap K_n} \{f_\alpha(y)\}, \{f_\alpha(h_n(\overline{x_n}))\} \right) \leq c^n \cdot \text{diam}(K)$$

for all $n \in \mathbb{N}$, and, via 5, the justification of the Claim is done.

Note that

$$\bigcup_{x \in K} A_{\{x\}} \stackrel{\text{Remark 7}}{\in} P_{cp}(X).$$

Now, as

$$\begin{aligned} h \left(G_S^{[n]}(K), \bigcup_{x \in K} A_{\{x\}} \right) &\leq h \left(G_S^{[n]}(K), \bigcup_{x \in K} G_S^{[n]}(\{x\}) \right) \\ &+ h \left(\bigcup_{x \in K} G_S^{[n]}(\{x\}), \bigcup_{x \in K} A_{\{x\}} \right) \stackrel{\text{Proposition 1}}{\leq} \\ &\leq h \left(G_S^{[n]}(K), \bigcup_{x \in K} G_S^{[n]}(\{x\}) \right) \\ &+ \sup_{x \in K} h \left(G_S^{[n]}(\{x\}), A_{\{x\}} \right) \stackrel{\text{Theorem 4}}{\leq} \\ &\leq h \left(G_S^{[n]}(K), \bigcup_{x \in K} G_S^{[n]}(\{x\}) \right) + \frac{c^n}{1-c} \cdot C_K, \end{aligned}$$

for every $n \in \mathbb{N}$, via the Claim, and the fact that

$$h \left(A_K, \bigcup_{x \in K} A_{\{x\}} \right) = \lim_{n \rightarrow \infty} h \left(G_S^{[n]}(K), \bigcup_{x \in K} A_{\{x\}} \right),$$

we deduce that

$$h \left(A_K, \bigcup_{x \in K} A_{\{x\}} \right) = 0,$$

and the conclusion follows. \square

Theorem 6. The function $f : P_{cp}(X) \rightarrow P_{cp}(X)$, given by

$$f(K) = A_K,$$

for every $K \in P_{cp}(X)$, is continuous.

Proof. We shall prove that f is sequentially continuous. To this end, we consider $(K_n)_{n \in \mathbb{N}} \subseteq P_{cp}(X)$ and $K \in P_{cp}(X)$ such that $\lim_{n \rightarrow \infty} h(K_n, K) = 0$. We will prove that

$$\lim_{n \rightarrow \infty} h(f(K_n), f(K)) = 0,$$

$$\text{i.e. } \lim_{n \rightarrow \infty} h(A_{K_n}, A_K) = 0.$$

Let us consider $\varepsilon > 0$ fixed, but arbitrarily chosen. Since, in view of Lemma 5, t is uniformly continuous on $K \cup \bigcup_{n \in \mathbb{N}} K_n \stackrel{\text{Proposition 2}}{\in} P_{cp}(X)$, there exists $\delta_\varepsilon > 0$ such that

$$h(t(x), t(y)) < \varepsilon, \tag{9}$$

for every $x, y \in K \cup \bigcup_{n \in \mathbb{N}} K_n$ having the property that $d(x, y) < \delta_\varepsilon$.

Moreover, as $\lim_{n \rightarrow \infty} h(K_n, K) = 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$h(K_n, K) < \delta_\varepsilon, \tag{10}$$

for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

The following Claim will finish the proof.

Claim. We have

$$h(A_{K_n}, A_K) \leq \varepsilon,$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Justification of Claim: Let us consider a fixed, but arbitrarily chosen $n \in \mathbb{N}$ such that $n \geq n_\varepsilon$.

For every $u \in A_{K_n} \stackrel{\text{Theorem 5}}{=} \bigcup_{x \in K_n} t(x)$, there exists $x_n \in K_n$ such that $u \in t(x_n)$.

Since

$$d(x_n, K) \leq D(K_n, K) \leq h(K_n, K) \stackrel{(10)}{<} \delta_\varepsilon,$$

there exists $y_n \in K$ such that

$$d(x_n, y_n) < \delta_\varepsilon, \tag{11}$$

so

$$d(u, t(y_n)) \leq D(t(x_n), t(y_n)) \leq h(t(x_n), t(y_n)) \stackrel{(9) \& (11)}{<} \varepsilon,$$

and consequently we can find $v \in t(y_n) \subseteq \bigcup_{x \in K} A_{\{x\}} \stackrel{\text{Theorem 5}}{=} A_K$ such that

$$d(u, v) < \varepsilon. \tag{12}$$

Therefore we see that

$$d(u, A_K) \leq d(u, v) \stackrel{(12)}{<} \varepsilon.$$

As $u \in A_{K_n}$ was arbitrarily chosen, it follows that

$$D(A_{K_n}, A_K) \leq \varepsilon.$$

One can prove similarly that

$$D(A_K, A_{K_n}) \leq \varepsilon,$$

hence the justification of the Claim is complete. \square

Theorem 7. For every $K \in P_{cp}(X)$, we have

$$G_S(A_K) = A_K,$$

so $G_S : P_{cp}(X) \rightarrow P_{cp}(X)$ is weakly Picard.

Proof. Let $K \in P_{cp}(X)$ be fixed, but arbitrarily chosen.

In view of Lemma 3, we have

$$A_{\{y\}} \subseteq [y],$$

hence, we deduce that

$$A_{\{y\}}^x = A_{\{y\}} \cap [x] = \begin{cases} A_{\{y\}}, & \text{if } [x] = [y] \\ \emptyset, & \text{if } [x] \neq [y] \end{cases} \stackrel{\text{Lemma 4}}{=} \begin{cases} A_{\{x\}}, & \text{if } [x] = [y] \\ \emptyset, & \text{if } [x] \neq [y] \end{cases}, \tag{13}$$

for any $x, y \in X$.

We have

$$G_S(A_{\{x\}}) = \bigcup_{i \in I, y \in X} f_i(A_{\{x\}}^y \times A_{\{x\}}^y) \stackrel{(13)}{=} \bigcup_{i \in I} f_i(A_{\{x\}} \times A_{\{x\}}). \tag{14}$$

Claim.

$$G_S(A_{\{x\}}) = A_{\{x\}},$$

for any $x \in X$.

Justification of Claim: Since

$$\begin{aligned} &f_i \left(G_S^{[n]}(\{x\}) \times G_S^{[n]}(\{x\}) \right) \\ &\stackrel{(7)}{=} f_i \left(\left(\bigcup_{\alpha \in nI} \{f_\alpha(h_n(\overline{x_n}))\} \right) \times \left(\bigcup_{\alpha \in nI} \{f_\alpha(h_n(\overline{x_n}))\} \right) \right) = \\ &= \bigcup_{\alpha, \beta \in nI} f_i(f_\alpha(h_n(\overline{x_n})), f_\beta(h_n(\overline{x_n}))) = \bigcup_{\gamma \in n+1I, \gamma_1=i} f_\gamma(h_{n+1}(\overline{x_{n+1}})), \end{aligned}$$

for every $i \in I$ and $n \in \mathbb{N}$, by passing to limit as $n \rightarrow \infty$, we get

$$f_i(A_{\{x\}} \times A_{\{x\}}) = \lim_{n \rightarrow \infty} \bigcup_{\gamma \in n+1I, \gamma_1=i} f_\gamma(h_{n+1}(\overline{x_{n+1}})), \tag{15}$$

for every $i \in I$.

Thus

$$\begin{aligned} G_S(A_{\{x\}}) &\stackrel{(14)}{=} \bigcup_{i \in I} f_i(A_{\{x\}} \times A_{\{x\}}) \stackrel{(15)}{=} \bigcup_{i \in I} \lim_{n \rightarrow \infty} \bigcup_{\gamma \in_{n+1} I, \gamma_1=i} f_\gamma(h_{n+1}(\overline{x_{n+1}})) = \\ &= \lim_{n \rightarrow \infty} \bigcup_{\gamma \in_{n+1} I} f_\gamma(h_{n+1}(\overline{x_{n+1}})) \stackrel{(7)}{=} \lim_{n \rightarrow \infty} G_S^{[n]}(\{x\}) = A_{\{x\}}, \end{aligned}$$

for any $x \in X$ and the justification of the Claim is done.

Finally, since

$$A_K^x = A_K \cap [x] \stackrel{\text{Theorem 5}}{=} \bigcup_{y \in K} (A_{\{y\}} \cap [x]) = \begin{cases} A_{\{x\}}, & \text{if } [x] \in [K] \\ \emptyset, & \text{if } [x] \notin [K] \end{cases},$$

we have

$$\begin{aligned} G_S(A_K) &= \bigcup_{i \in I, x \in X} f_i(A_K^x \times A_K^x) = \bigcup_{i \in I, x \in X, [x] \in [K]} f_i(A_{\{x\}} \times A_{\{x\}}) \\ &\stackrel{(14)}{=} \bigcup_{x \in X, [x] \in [K]} G_S(A_{\{x\}}) \stackrel{\text{Claim}}{=} \\ &= \bigcup_{x \in X, [x] \in [K]} A_{\{x\}} \stackrel{\text{Lemma 4}}{=} \bigcup_{x \in K} A_{\{x\}} \stackrel{\text{Theorem 5}}{=} A_K. \quad \square \end{aligned}$$

Lemma 6. Let $K_1, K_2 \in P_{cp}(X)$ be such that $[K_1] = [K_2]$ and set $K_3 = F_{S,R}(K_1, K_2)$. Then $[K_3] = [K_1] = [K_2]$.

Proof. On the one hand, we have $[K_3] \subseteq [K_1]$.

Indeed, for $c \in K_3$ there exist $i \in I, x \in X, a \in K_1, b \in K_2$ such that $a, b \in [x]$ and $c = f_i(a, b)$. Then $c \in [x] = [a] \in [K_1]$, so $[c] \in [K_1]$.

On the other hand, we have $[K_1] \subseteq [K_3]$.

Indeed, for $a \in K_1$, since $K_2 \cap [a] \neq \emptyset$, there exists $b \in K_2 \cap [a]$. Therefore, considering $c = f_i(a, b) \in [a]$, for some $i \in I$, we have $a \in [c] = [a] \in [K_3]$. \square

Proposition 4. (\tilde{K}, h) is complete for every $K \in P_{cp}(X)$.

Proof. Since $P_{cp}(X)$ is complete, it suffices to prove that for every $(H_n)_{n \in \mathbb{N}} \subseteq \tilde{K}$ and $H \in P_{cp}(X)$ such that $\lim_{n \rightarrow \infty} h(H_n, H) = 0$, we have $H \overset{R}{\sim} K$, i.e., in view of Lemma 1,

$$K \cap [x] \neq \emptyset \Leftrightarrow H \cap [x] \neq \emptyset,$$

for any $x \in X$.

$\varepsilon \Rightarrow \varepsilon$ Let us fix $x \in X$ such that $K \cap [x] \neq \emptyset$. Since $[H_n] = [K]$, there exists $x_n \in H_n \cap [x]$ for each $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, in view of Remark 2, there exists $y_n \in H$ such that

$$d(x_n, y_n) < h(H_n, H) + \frac{1}{n}.$$

Since H is compact, there exists $y \in H$ and $(y_{n_k})_k \subseteq (y_n)_n$ such that $\lim_{k \rightarrow \infty} y_{n_k} = y$. Then, since

$$d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y) < h(H_{n_k}, H) + \frac{1}{n_k} + d(y_{n_k}, y),$$

for all $k \in \mathbb{N}$, we deduce that $\lim_{k \rightarrow \infty} x_{n_k} = y \in [x]$. Therefore $y \in H \cap [x]$, so $H \cap [x] \neq \emptyset$.

$\varepsilon \Leftarrow \varepsilon$ Let $x \in X$ be such that $H \cap [x] \neq \emptyset$. Then there exists $y \in H \cap [x]$, so $[x] = [y]$. For each $n \in \mathbb{N}$ we find $y_n \in H_n$ such that

$$d(y_n, y) < h(H_n, H) + \frac{1}{n},$$

hence $\lim_{n \rightarrow \infty} y_n = y$. Since $[H_n] = [K]$, there exists $z_n \in K$ such that $[z_n] = [y_n]$ for each $n \in \mathbb{N}$. As K is compact, there exist $(z_{n_k})_{k \in \mathbb{N}} \subseteq (z_n)_{n \in \mathbb{N}}$ and $z \in K$ such that $\lim_{k \rightarrow \infty} z_{n_k} = z$. Using the fact that R is closed and since

$$\lim_{k \rightarrow \infty} y_{n_k} = y, \lim_{k \rightarrow \infty} z_{n_k} = z \text{ and } [y_{n_k}] = [z_{n_k}],$$

for all $k \in \mathbb{N}$, taking into account Remark 4, we deduce that $[x] = [y] = [z]$, so $z \in K \cap [x]$, therefore $K \cap [x] \neq \emptyset$. \square

Theorem 8. We have that

$$[A_K] = [K]$$

for any $K \in P_{cp}(X)$.

Proof. Using Lemma 6, we deduce that

$$[K] = [G_S(K)] = [G_S^{[2]}(K)] = \dots = [G_S^{[n]}(K)],$$

for any $n \in \mathbb{N}$, and, in view of Proposition 4, we conclude that

$$[A_K] = [K],$$

for any $K \in P_{cp}(X)$. \square

Proposition 5. For any $K_1, K_2 \in P_{cp}(X)$, the following statements are equivalent:

- (i) $A_{K_1} = A_{K_2}$;
- (ii) $[K_1] = [K_2]$.

Proof. (i) \Rightarrow (ii) Let $K_1, K_2 \in P_{cp}(X)$ be such that $A_{K_1} = A_{K_2}$, i.e., in view of Theorem 5, $\bigcup_{x \in K_1} A_{\{x\}} = \bigcup_{x \in K_2} A_{\{x\}}$. Let $x \in K_1$. Then $A_{\{x\}} \subseteq \bigcup_{x \in K_2} A_{\{x\}}$, which implies, via Lemmas 3 and 4, that there exists $y \in K_2$ such that $[x] = [y]$. Hence we proved that $[x] \in [K_2]$. Since $x \in K_1$ was arbitrarily chosen, we conclude that $[K_1] \subseteq [K_2]$. The reverse inclusion can be proved in a similar way. (ii) \Rightarrow (i) Suppose now that $[K_1] = [K_2]$ and let $x \in K_1$. Then there exists $y \in K_2$ such that $[x] = [y]$. Making use of Lemma 4, we infer that $A_{\{x\}} = A_{\{y\}}$, therefore, in view of Theorem 5, we have that $A_{\{x\}} \subseteq A_{K_2}$. Since $x \in K_1$ was arbitrarily chosen, based on Theorem 5, we conclude that $A_{K_1} \subseteq A_{K_2}$. The reverse inclusion can be proved in a similar way. \square

For $x \in X$ let us consider the GIFS $S_x = (([x], d), (f_i^x)_{i \in I})$, where $f_i^x : [x] \times [x] \rightarrow [x]$ is given by

$$f_i^x(u, v) = f_i(u, v),$$

for all $u, v \in [x]$ and $i \in I$.

Note that, based on Remark 5, the functions f_i^x are well defined.

Let us consider the fractal operator associated with S_x , namely $F_{S_x} : P_{cp}([x]) \times P_{cp}([x]) \rightarrow P_{cp}([x])$ given by

$$F_{S_x}(K, K) = \bigcup_{i \in I} f_i^x(K \times K),$$

for all $K \in P_{cp}([x])$.

According to the results presented in Subsection 2.3, there exists a unique $A_{S_x} \in P_{cp}([x])$ such that

$$F_{S_x}(A_{S_x}, A_{S_x}) = A_{S_x},$$

and, for every $K_1, K_2 \in P_{cp}([x])$ and $n \in \mathbb{N}$, we have

$$h(K_n, A_{S_x}) \leq 2 \frac{c^{\lfloor \frac{n}{2} \rfloor}}{1-c} \cdot \text{diam}(K_1 \cup K_2 \cup K_3), \tag{16}$$

where the sequence $(K_n)_{n \in \mathbb{N}}$ is defined by

$$K_{n+2} = F_{S_x}(K_{n+1}, K_n)$$

for every $n \in \mathbb{N}$, so

$$A_{S_x} = \lim_{n \rightarrow \infty} K_n.$$

Note that

$$A_{\{x\}} = \lim_{n \rightarrow \infty} G_S^{[n]}(\{x\}) \stackrel{(7)}{=} A_{S_x}, \tag{17}$$

for every $x \in X$.

Let us now consider $K_1, K_2 \in P_{cp}(X)$ such that $K_1 \overset{R}{\sim} K_2$, i.e. $[K_1] = [K_2]$, and the sequence $(K_n)_{n \in \mathbb{N}}$ defined by

$$K_{n+2} = F_{S,R}(K_{n+1}, K_n),$$

for every $n \in \mathbb{N}$.

Note that, in view of Lemma 6, we have

$$[K_1] = [K_2] = \dots = [K_n], \tag{18}$$

for every $n \in \mathbb{N}$, so the sequence $(K_n)_{n \in \mathbb{N}}$ is well-defined.

Lemma 7. We have

$$K_n = \bigcup_{x \in K_1} K_n^x \text{ and } K_{n+2}^x = F_{S_x}(K_{n+1}^x, K_n^x),$$

for every $n \in \mathbb{N}$ and $x \in K_1$.

Proof. Let us fix $n \in \mathbb{N}$.

On the one hand, we have

$$\bigcup_{x \in K_1} K_n^x = \bigcup_{x \in K_1} (K_n \cap [x]) \subseteq K_n.$$

On the other hand, we have

$$K_n \subseteq \bigcup_{x \in K_1} K_n^x.$$

Indeed, let us consider $y \in K_n$. Since $[K_1] = [K_n]$, there exists $x \in K_1$ such that $y \in [x]$. Therefore $y \in K_n^x \subseteq \bigcup_{x \in K_1} K_n^x$.

Hence $K_n = \bigcup_{x \in K_1} K_n^x$.

Next, observe that for any $x \in K_1$ we have

$$\begin{aligned} K_{n+2}^x &= K_{n+2} \cap [x] = F_{S_x}(K_{n+1}, K_n) \cap [x] \\ &= \left(\bigcup_{i \in I, y \in K_1} f_i(K_{n+1}^y \times K_n^y) \right) \cap [x] = \\ &= \bigcup_{i \in I} f_i(K_{n+1}^x \times K_n^x) = \bigcup_{i \in I} f_i^x(K_{n+1}^x \times K_n^x) = F_{S_x}(K_{n+1}^x, K_n^x). \quad \square \end{aligned}$$

Lemma 8. $(K_n)_{n \in \mathbb{N}}$ is convergent.

Proof. It suffices to show that $(K_n)_{n \in \mathbb{N}}$ is Cauchy.

Let us first note that

$$\begin{aligned} h(K_{n+3}, K_{n+2}) &\stackrel{\text{Lemma 7}}{=} h\left(\bigcup_{x \in K_1} K_{n+3}^x, \bigcup_{x \in K_1} K_{n+2}^x\right) \leq \sup_{x \in K_1} h(K_{n+3}^x, K_{n+2}^x) \leq \\ &\leq \sup_{x \in K_1} \left(h(K_{n+3}^x, A_{S_x}) + h(A_{S_x}, K_{n+2}^x) \right) \stackrel{(16)}{\leq} \\ &\leq \frac{2}{1-c} \left(c^{\lfloor \frac{n+3}{2} \rfloor} + c^{\lfloor \frac{n+2}{2} \rfloor} \right) \cdot \text{diam}(K_1 \cup K_2 \cup K_3) \leq \\ &\leq \frac{4}{1-c} c^{\lfloor \frac{n}{2} \rfloor} \cdot \text{diam}(K_1 \cup K_2 \cup K_3), \end{aligned} \tag{19}$$

for every $n \in \mathbb{N}$.

Hence

$$\begin{aligned} h(K_{n+p}, K_{n+2}) &\leq h(K_{n+p}, K_{n+p-1}) + \dots + h(K_{n+3}, K_{n+2}) \stackrel{(19)}{\leq} \\ &\leq \frac{4}{1-c} \left(c^{\lfloor \frac{n+p-3}{2} \rfloor} + \dots + c^{\lfloor \frac{n}{2} \rfloor} \right) \cdot \text{diam}(K_1 \cup K_2 \cup K_3) < \\ &< \frac{4}{1-c} \left(c^{\frac{n+p-3}{2}-1} + \dots + c^{\frac{n}{2}-1} \right) \cdot \text{diam}(K_1 \cup K_2 \cup K_3) = \\ &= \frac{4}{1-c} c^{\frac{n}{2}-1} \frac{1-c^{p-2}}{1-c^{\frac{1}{2}}} \cdot \text{diam}(K_1 \cup K_2 \cup K_3) < \\ &< \frac{4}{(1-c)\left(1-c^{\frac{1}{2}}\right)} c^{\frac{n}{2}-1} \cdot \text{diam}(K_1 \cup K_2 \cup K_3), \end{aligned}$$

for any $n, p \in \mathbb{N}$, and this means that $(K_n)_{n \in \mathbb{N}}$ is Cauchy. \square

Theorem 9. We have

$$\lim_{n \rightarrow \infty} K_n = A_{K_1} = A_{K_2}.$$

Proof. Let us first note that since $[K_1] = [K_2]$, we can assert that

$$A_{K_1} \stackrel{\text{Theorem 5}}{=} \bigcup_{x \in K_1} A_{[x]} \stackrel{\text{Lemma 4}}{=} \bigcup_{x \in K_2} A_{[x]} \stackrel{\text{Theorem 5}}{=} A_{K_2}.$$

Now, according to Lemma 8, there exists $A_{K_1, K_2} \in P_{cp}(X)$ such that

$$\lim_{n \rightarrow \infty} K_n = A_{K_1, K_2}.$$

The proof will be complete once we show that $A_{K_1, K_2} = A_{K_1}$.

Let us note that

$$\begin{aligned} h(K_n, A_{K_1}) &\stackrel{\text{Theorem 5}}{=} h\left(K_n, \bigcup_{x \in K_1} A_{[x]}\right) \\ &\stackrel{\text{Lemma 7}}{=} h\left(\bigcup_{x \in K_1} K_n^x, \bigcup_{x \in K_1} A_{[x]}\right) \stackrel{\text{Proposition 1}}{\leq} \\ &\leq \sup_{x \in K_1} h(K_n^x, A_{[x]}) \stackrel{(17)}{=} \sup_{x \in K_1} h(K_n^x, A_{S_x}) \\ &\stackrel{(16)}{\leq} 2 \frac{c^{\lfloor \frac{n}{2} \rfloor}}{1-c} \cdot \text{diam}(K_1 \cup K_2 \cup K_3), \end{aligned} \tag{20}$$

for any $n \in \mathbb{N}$.

By taking limits as $n \rightarrow \infty$ in (20), we deduce that

$$\lim_{n \rightarrow \infty} K_n = A_{K_1},$$

so

$$A_{K_1, K_2} = A_{K_1} = A_{K_2}. \quad \square$$

5. Examples

In this section we present four examples which illustrate our results. The first example shows us that RGIFSs are generalizations of the classical GIFSs, in the sense that every GIFS is a RGIFS when one considers a particular equivalence relation, while the other three examples highlight the fact that the generalization is effective, since the fractal operator in these three cases is not Picard, but weakly Picard, because it admits more than one fixed point.

I. Let us consider the RGIFS $S = ((X, d), \{f_i\}_{i \in I}, R)$, where, for $x, y \in X$, we consider xRy if and only if $d(x, y) < \infty$. It is straightforward to verify that R is an equivalence relation on X . For $x \in X$, we have $[x] = X$, therefore R is a closed equivalence relation and $R = X \times X$.

Note that condition (i) from Definition 6 implies the existence of $c \in [0, 1)$ such that

$$d(f_i(x_1, y_1), f_i(x_2, y_2)) \leq c \cdot \max\{d(x_1, x_2), d(y_1, y_2)\},$$

for any $x_1, y_1, x_2, y_2 \in X$ and $i \in I$, while condition (ii) is trivially satisfied.

In view of Proposition 5, the attractor of S is unique.

Hence, in this case we obtain the classical GIFS of order 2.

II. Let us consider two complete metric spaces (X, d) , (Y, ρ) and a finite, nonempty set of indices I .

For each $i \in I$ we consider a continuous function $g_i : X^2 \rightarrow X$ such that

$$g_i(x, x) = x,$$

for every $x \in X$.

For each $i \in I$ let us also consider a continuous function $H_i : X \times Y^2 \rightarrow Y$ such that for every $x \in X$, the function $h_{i,x} : Y^2 \rightarrow Y$, given by

$$h_{i,x}(y_1, y_2) = H_i(x, y_1, y_2)$$

for all $y_1, y_2 \in Y$, has the property that there exists $c \in [0, 1)$ such that

$$\rho(h_{i,x}(u_1, u_2), h_{i,x}(v_1, v_2)) \leq c \cdot \max\{\rho(u_1, v_1), \rho(u_2, v_2)\},$$

for all $u_1, u_2, v_1, v_2 \in Y$.

For every $x \in X$ let us denote by A^x the attractor of $S_x = (Y, \rho, (h_{i,x})_{i \in I})$.

On $X \times Y$ we consider the following closed equivalence relation:

$$(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1 = x_2.$$

For each $i \in I$, let us define the function $f_i : (X \times Y)^2 \rightarrow X \times Y$, given by

$$f_i((x_1, y_1), (x_2, y_2)) = (g_i(x_1, x_2), h_{i,g_i(x_1, x_2)}(y_1, y_2)),$$

for each $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Then $S = (X \times Y, (f_i)_{i \in I}, R)$ is a RGIFS.

Indeed, let us verify the two conditions from Definition 6.

In order to check the first condition, let us consider $(x_1, y_1), (x_1, u_1), (x_2, y_2), (x_2, u_2) \in X \times Y$ and note that

$$\begin{aligned} d_{X \times Y}(f_i((x_1, y_1), (x_2, y_2)), f_i((x_1, u_1), (x_2, u_2))) &= \\ &= d_{X \times Y}((g_i(x_1, x_2), h_{i,g_i(x_1, x_2)}(y_1, y_2)), (g_i(x_1, x_2), h_{i,g_i(x_1, x_2)}(u_1, u_2))) = \\ &= \rho(h_{i,g_i(x_1, x_2)}(y_1, y_2), h_{i,g_i(x_1, x_2)}(u_1, u_2)) \leq c \cdot \max\{\rho(y_1, u_1), \rho(y_2, u_2)\} = \\ &= c \cdot \max\{d_{X \times Y}((x_1, y_1), (x_1, u_1)), d_{X \times Y}((x_2, y_2), (x_2, u_2))\}. \end{aligned}$$

As for the second condition, let $(x, y_1), (x, y_2) \in X \times Y$ and note that

$$\begin{aligned} f_i((x, y_1), (x, y_2)) &= (g_i(x, x), h_{i,g_i(x, x)}(y_1, y_2)) \\ &= (x, h_{i,x}(y_1, y_2)) R(x, y_1)R(x, y_2) \end{aligned}$$

for all $i \in I$.

Now, let $x \in X$ and note that

$$G_S^{[n]}(\{(x, y)\}) = (x, F_S^{[n]}(\{(x, y)\})),$$

for every $n \in \mathbb{N}$ and $y \in Y$.

Therefore, we have

$$A_{\{(x, y)\}} = \{x\} \times A^x,$$

for every $(x, y) \in X \times Y$, which, based on Theorem 5, implies that

$$A_{\{x\} \times K} = \{x\} \times A^x,$$

for any $K \in P_{cp}(Y)$, and

$$A_{H \times K} = \bigcup_{x \in H} (\{x\} \times A^x)$$

for any $H \in P_{cp}(X)$ and $K \in P_{cp}(Y)$.

Let us mention that if $(X, d) = (Y, \rho) = (\mathbb{R}, |\cdot|)$, for any $i \in I$, we can consider $g_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$g_i(x_1, x_2) = \frac{x_1 + x_2}{2}$$

for any $x_1, x_2 \in \mathbb{R}$ and $H_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$H_i(x, y_1, y_2) = \frac{y_1 + y_2}{4} + a_i x$$

for any $x, y_1, y_2 \in \mathbb{R}$, where $a_i \in \mathbb{R}$.

If $(X, d) = ([0, \infty), |\cdot|)$, $(Y, \rho) = (\mathbb{R}, |\cdot|)$ for any $i \in I$, we can consider $g_i : [0, \infty)^2 \rightarrow [0, \infty)$ given by

$$g_i(x_1, x_2) = \sqrt{x_1 x_2},$$

for any $x_1, x_2 \geq 0$ and $H_i : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$H_i(x, y_1, y_2) = \frac{y_1 + y_2}{4} + a_i x,$$

for any $x \geq 0, y_1, y_2 \in \mathbb{R}$, where $a_i \in \mathbb{R}$.

III. Let us consider the triplet $S = ((X, d), \{f_i\}_{i \in I}, R)$, where

- $(X, d) = (\mathbb{R}^2, d)$, d being the Euclidean distance
- $I = \{1, 2\}$

• $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by

$$f_1((x_1, y_1), (x_2, y_2)) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{4}\right)$$

and

$$f_2((x_1, y_1), (x_2, y_2)) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{4} + \frac{\alpha(x_1, x_2)}{2}\right),$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$, where $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

• $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 = x_2$ for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Then $S = ((X, d), \{f_i\}_{i \in I}, R)$ is a RGIFS.

It is straightforward to verify that R is an equivalence relation on \mathbb{R}^2 , and, since

$$[(x, y)] = \{(u, v) \in \mathbb{R}^2 : x = u\},$$

for all $x, y \in \mathbb{R}$, we infer that it is closed.

For $x, y, z, w_1, w_2 \in \mathbb{R}$ we have

$$\begin{aligned} d(f_1((x, y_1), (z, w_1)), f_1((x, y_2), (z, w_2))) &= \\ &= d\left(\left(\frac{x+z}{2}, \frac{y_1+w_1}{4}\right), \left(\frac{x+z}{2}, \frac{y_2+w_2}{4}\right)\right) = \\ &= \left|\frac{y_1+w_1}{4} - \frac{y_2+w_2}{4}\right| \leq \frac{1}{4}|y_1 - y_2| + \frac{1}{4}|w_1 - w_2| \\ &\leq \frac{1}{2} \max\{|y_1 - y_2|, |w_1 - w_2|\} = \\ &= \frac{1}{2} \max\{d((x, y_1), (x, y_2)), d((z, w_1), (z, w_2))\} \end{aligned}$$

and, similarly,

$$\begin{aligned} d(f_2((x, y_1), (z, w_1)), f_2((x, y_2), (z, w_2))) &= \\ &= d\left(\left(\frac{x+z}{2}, \frac{y_1+w_1}{4} + \frac{\alpha(x, z)}{2}\right), \left(\frac{x+z}{2}, \frac{y_2+w_2}{4} + \frac{\alpha(x, z)}{2}\right)\right) = \\ &= \left|\frac{y_1+w_1}{4} - \frac{y_2+w_2}{4}\right| \leq \frac{1}{2} \max\{d((x, y_1), (x, y_2)), d((z, w_1), (z, w_2))\}. \end{aligned}$$

Thus we deduce that condition (i) from Definition 6 is fulfilled for $c = \frac{1}{2}$. As for condition (ii) from Definition 6, note that

$$f_1((x, y_1), (x, y_2)) = \left(x, \frac{y_1 + y_2}{4}\right) R(x, y)$$

and

$$f_2((x, y_1), (x, y_2)) = \left(x, \frac{y_1 + y_2}{4} + \frac{\alpha(x, x)}{2}\right) R(x, y),$$

for all $x, y, y_1, y_2 \in \mathbb{R}$.

For $x \in \mathbb{R}$ we consider $S_x = ((\mathbb{R}, |\cdot|), (h_{i,x})_{i \in \{1,2\}})$, where $h_{1,x}, h_{2,x} : \mathbb{R}^2 \rightarrow \mathbb{R}$ are given by

$$h_{1,x}(y_1, y_2) = \frac{y_1 + y_2}{4}$$

and

$$h_{2,x}(y_1, y_2) = \frac{y_1 + y_2}{4} + \frac{\alpha(x, x)}{2},$$

for every $y_1, y_2 \in \mathbb{R}$.

Since $[0, \alpha(x, x)] \in P_{cp}(\mathbb{R})$ and

$$\begin{aligned} F_{S_x}([0, \alpha(x, x)], [0, \alpha(x, x)]) &= \\ &= h_{1,x}([0, \alpha(x, x)] \times [0, \alpha(x, x)]) \cup h_{2,x}([0, \alpha(x, x)] \times [0, \alpha(x, x)]) = \\ &= \left[0, \frac{\alpha(x, x)}{2}\right] \cup \left[\frac{\alpha(x, x)}{2}, \alpha(x, x)\right] = \\ &= [0, \alpha(x, x)], \end{aligned}$$

we infer that

$$A_{S_x} = [0, \alpha(x, x)].$$

As

$$G_S^{[n]}(\{(x, y)\}) = \{x\} \times F_{S_x}^{[n]}(\{y\}, \{y\}),$$

for every $x, y \in \mathbb{R}$ and every $n \in \mathbb{N}$, by passing to limit as $n \rightarrow \infty$, we conclude that

$$A_{\{(x,y)\}} = \{x\} \times [0, \alpha(x, x)]$$

for every $x, y \in \mathbb{R}$.

In view of Theorem 5,

$$A_{\{x\} \times K} = \bigcup_{y \in K} A_{\{(x,y)\}} = \{x\} \times [0, \alpha(x, x)]$$

for any $x \in \mathbb{R}, K \in P_{cp}(\mathbb{R})$, so

$$A_{H \times K} = \bigcup_{x \in H} (\{x\} \times [0, \alpha(x, x)])$$

for any $H, K \in P_{cp}(\mathbb{R})$.

IV. Let us consider the triplet $S = ((X, d), \{f_i\}_{i \in I}, R)$, where

- $(X, d) = (\mathbb{R}^2, d)$, d being the Euclidean distance
- $I = \{1, 2\}$
- $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by

$$f_1((x_1, y_1), (x_2, y_2)) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{6} \right)$$

and

$$f_2((x_1, y_1), (x_2, y_2)) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{6} + \frac{\alpha(x_1, x_2)}{2} \right)$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$, where $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

- $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 = x_2$, for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Similarly to the previous example, it can be shown that $S = ((X, d), \{f_i\}_{i \in I}, R)$ is a RGIFS.

As in the previous example, for every $x \in \mathbb{R}$ we consider $S_x = ((\mathbb{R}, |\cdot|), (h_{i,x})_{i \in \{1,2\}})$, where $h_{1,x}, h_{2,x} : \mathbb{R}^2 \rightarrow \mathbb{R}$ are given by

$$h_{1,x}(y_1, y_2) = \frac{y_1 + y_2}{6}$$

and

$$h_{2,x}(y_1, y_2) = \frac{y_1 + y_2}{6} + \frac{\alpha(x, x)}{2}$$

for every $y_1, y_2 \in \mathbb{R}$ and we get that

$$A_{S_x} = \{x\} \times \left(\left[0, \frac{\alpha(x, x)}{4} \right] \cup \left[\frac{\alpha(x, x)}{2}, \frac{3\alpha(x, x)}{4} \right] \right).$$

Consequently we have

$$A_{H \times K} = \bigcup_{x \in H} \left(\{x\} \times \left(\left[0, \frac{\alpha(x, x)}{4} \right] \cup \left[\frac{\alpha(x, x)}{2}, \frac{3\alpha(x, x)}{4} \right] \right) \right),$$

for every $H, K \in P_{cp}(\mathbb{R})$.

6. Conclusions

In this paper, we studied a new class of generalized iterated function systems, in which we imposed a contractive condition upon the functions in the system that involves an equivalence relation defined on the underlying metric space. We called such systems relational generalized iterated function systems (RGIFSs) and proved through some examples that they constitute a real generalization of the usual generalized iterated function systems (GIFSs), in the sense that a particular equivalence relation in the frame of a RGIFS yields the classical contractive condition corresponding to GIFSs and there exist RGIFSs where the associated fractal operator is only weakly Picard, which means that they are not GIFSs.

CRediT authorship contribution statement

Izabella Abraham: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Conceptualization. **Radu Miculescu:** Writing – review & editing, Writing – original draft, Validation, Investigation, Conceptualization. **Alexandru Mihail:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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