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ORIGINAL PAPER

On the fractal operator of a mixed possibly infinite iterated function system

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Abstract

In this paper we introduce a new class of iterated function systems. More precisely, we study the fractal operator associated with a mixed possibly infinite iterated function system (briefly mIIFS). Such a system is a possibly infinite iterated function system (i.e. a possibly infinite family of Banach contractions on a complete metric space, satisfying some extra conditions) enriched with an orbital possibly infinite iterated function system (i.e. a possible infinite family of nonexpansive functions which need not be Banach contractions on the entire previously mentioned complete metric space, but just on the orbits of the space's elements). Our main result states that the fractal operator associated with an mIIFS is a weakly Picard operator. Its fixed points are called attractors of the system. We present concrete examples of mIIFSs and graphical representations of certain attractors' approximates.

Keywords Possibly infinite iterated function system · Orbital possibly infinite iterated function system · Mixed possibly infinite iterated function system · Fractal operator · Weakly Picard operator · Attractor

Mathematics Subject Classification 28A80 · 37C70 · 54H20

1 Introduction

Let us recall that an iterated function system (for short IFS) consists of a complete metric space (X, d) and a finite family of continuous functions $f_i : X \to X$, $i \in I$. Then the system $\mathcal{S} = ((X, d), (f_i)_{i\in I})$ generates the fractal operator $F_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$ given by $F_{\mathcal{S}}(K) = \bigcup_{i\in I} f_i(K)$ for every $K \in P_{cp}(X)$, where $P_{cp}(X)$ is the hyperspace of all

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nonempty and compact subsets of X. In 1981, Hutchinson (see [12]) proved that, with respect to the Hausdorff–Pompeiu metric, $F_{\mathcal{S}}$ is a Banach contraction, provided that the functions f_i are Banach contractions. Its unique fixed point, denoted by $A_{\mathcal{S}}$, is called the attractor of $\mathcal{S}$. This terminology is based on the remark that $\lim_{n\to\infty}(\underbrace{F_{\mathcal{S}} \circ \cdots \circ F_{\mathcal{S}}}_{n \text{ times}})(K) = A_{\mathcal{S}}$ for every $K \in P_{cp}(X)$, which is valid in view of the fact that $F_{\mathcal{S}}$ is a Picard operator. Two excellent surveys on iterated function systems are [13, 16].

Barnsley and Demko (see [3]) realized that the above-mentioned result could play a central role in the modelling of partly random or chaotic phenomena and, consequently, the concept of iterated function system holds a central place in the theory of fractals and yields a large scale of applications in computer science (especially image compression), engineering sciences, economy, human anatomy, physics, forestry etc.

Hence, there is no surprise that, in the last decades, various extensions of the Hutchinson–Barnsley theory have been developed.

On the one hand, of particular interest from the point of view of the present paper, is the line of research that focuses on systems comprising arbitrary families (not necessarily finite) of functions. Some results in this direction can be found in [5, 8, 11, 14, 15, 19, 20, 23, 24, 37, 38].

On the other hand, also of special interest from the perspective of the present paper, is the line of research that concentrates on systems for which the functions f_i belong to classes of contractions which are larger than Banach's one. In this way, many generalizations of Banach contraction principle have natural analogues in iterated function systems theory. Let us mention some results in this direction.

In 1985, Hata (see [9, 10]) generalized Hutchinson's result by considering weak contractions. Later developments on this direction are due to Shiota (see [41]), Máté (see [18]), Georgescu and collaborators (see [6]) and Okamura (see [30]).

Edalat introduced in 1996 the concept of weakly hyperbolic IFS with finite parameter spaces (see Definition 1.1 from [4]) which was generalized by A. Arbieto and his collaborators (see [2]) by adopting the more general setting of arbitrary compact metric spaces as parameter spaces. Later, Melo proposed the concept of P-weakly hyperbolic iterated function systems on compact metric spaces (see Definition 1 from [22]) which generalizes the previous mentioned works.

In 2010, Sahu and his collaborators (see [35]) launched the concept of Kannan iterated function system. See also [7, 25, 33, 44] for systems comprising Reich, Chatterjea or Ćirić contractions.

Secelean (see [39]) started, in 2013, the investigation of iterated function systems consisting of F-contractions.

In 2017, Miculescu and Mihail (see [26]) introduced the concept of iterated function system consisting of convex contractions.

Pasupathi and his collaborators (see [31]), came up in 2020 with the notion of iterated function system consisting of cyclic contractions. For some other works in this direction see [1, 21, 32, 43].

Recently, Prithvi and Katiyar (see [34]) initiated the study of iterated function systems comprising interpolative δ-operators.

We make a point on the fact that for all the above-mentioned generalizations of the concept of iterated function system the corresponding fractal operator is a Picard operator.

In 2018, Miculescu et al. (see [27]) launched the study of iterated functions systems involving functions that satisfy Banach's orbital condition. See also [28, 36] (for the particular case of orbital contractive iterated function systems) and [29, 40].

We put the stress on the fact that the fractal operator associated with this type of iterated function systems is weakly Picard.

Very recently the study of the fractal operator associated with iterated function systems enriched with iterated function systems consisting of nonexpansive functions has aroused the interest of some researchers (see [17, 42]). More precisely, the object under investigation is $F_{\mathcal{S}}$, where $\mathcal{S} =((X, d), (f_i)_{i\in I\cup J})$ is an iterated function system such that f_i is a Banach contraction (or more generally, Matkowski contraction) for each $i \in I$ and $lip(f_i) \leq 1$ for each $i \in J$.

In this paper we combine the lines of research previously presented by studying the fractal operator associated with a mixed possibly infinite iterated function system (see Definition 2.8). Such a system is a possibly infinite iterated function system (i.e. a possibly infinite family of Banach contractions satisfying some extra conditions—see Definition 2.5) enriched with an orbital possibly infinite iterated function system (i.e. a possible infinite family of nonexpansive functions which are Banach contractions, not on the whole space, but just on the orbits of the space's elements). Theorem 3.7, which is the main result of this paper, states that the fractal operator associated with a mixed possibly infinite iterated function system is a weakly Picard operator. We call its fixed points attractors. In the last section, we present some concrete examples of such mixed finite iterated function systems. In addition, we provide some graphical representations of attractors' approximates.

2 Preliminaries

By $\mathbb{N}$ we mean the set $\{1, 2, \ldots\}$.

For a real number x, we denote by $[x]$ the integral part of x.

Let us recall the following basic result (called the generalized pigeonhole principle):

If N objects are placed in k boxes, then there exists at least one box having at least $\left[\frac{N}{k}\right]$ objects.

For the remainder of this section (X, d) will be a metric space.

Given a function $f: X \to X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we mean the composition of f by itself n times and by $f^{[0]}$ we understand Id_X.

For a function $f: X \to X$ and a subset A of X such that $f(A) \subseteq A$, by $f_{|A}$ we denote the map $f_{|A}: A \to A$, given by $f_{|A}(x) = f(x)$ for every $x \in A$.

By $lip(f)$ we denote the Lipschitz constant of a Lipschitz function $f: X \to X$.

For a subset A of X, by $\overline{A}$ we mean the closure of A.

Definition 2.1 A function $f: X \to X$ is called a weakly Picard operator if, for every $x \in X$, the sequence $(f^{[n]}(x))_{n\in\mathbb{N}}$ is convergent to a fixed point of f.

Let $A, B \subseteq X$ and $x \in X$. We shall use the following notation:

$$\begin{aligned}
P_b(X) &:= \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is bounded}\},\\
P_{b,cl}(X) &:= \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is closed and bounded}\},\\
P_{cp}(X) &:= \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\},\\
d(x, A) &:= \inf_{a\in A} d(x, a),\\
d(A, B) &:= \sup_{a\in A} d(a, B)
\end{aligned}$$

and

$$diam(A) := \sup_{x,y \in A} d(x, y).$$

Proposition 2.2 (see Proposition 2.1 from [24]) *The function* $h^*\colon P_b(X) \times P_b(X) \to [0, \infty)$*, described by*

$$h^*(A, B) = \max\{d(A, B), d(B, A)\},$$

for every $A, B \in P_b(X)$*, has the following properties:*

(i) *For every* $A, B \in P_b(X)$*, we have*

$$h^*(A, B) = h^*(\overline{A}, \overline{B}).$$

(ii) *For every* $\{A_i\}_{i \in I}$ *and* $\{B_i\}_{i \in I}$ *families of elements of* $P_b(X)$ *such that* $\bigcup_{i \in I} A_i \in P_b(X)$ *and* $\bigcup_{i \in I} B_i \in P_b(X)$*, we have*

$$h^*\left(\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i\right) \leq \sup_{i \in I} h^*(A_i, B_i).$$

Definition 2.3 The restriction of h^* to $P_{b,cl}(X) \times P_{b,cl}(X)$ is a metric called the Hausdorff–Pompeiu metric and it is denoted by h.

Lemma 2.4 *Let* A *and* B *be non-empty sets and let us consider* $R \subseteq A \times B$ *such that*

$$\forall a \in A \quad \exists b \in B \quad (a, b) \in R \qquad \text{and} \qquad \forall b \in B \quad \exists a \in A \quad (a, b) \in R.$$

Then

$$h^*\left(\bigcup_{a \in A} M_a, \bigcup_{b \in B} N_b\right) \leq \sup_{(a,b) \in R} h^*(M_a, N_b),$$

for every family $(M_a)_{a \in A}$ *and* $(N_b)_{b \in B}$ *of elements from* $P_b(X)$ *such that* $\bigcup_{a \in A} M_a$*,* $\bigcup_{b \in B} N_b \in P_b(X)$*.*

Proof We have

$$h^*\left(\bigcup_{a \in A} M_a, \bigcup_{b \in B} N_b\right) = h^*\left(\bigcup_{(a,b) \in R} M_a, \bigcup_{(a,b) \in R} N_b\right) \overset{\text{Proposition 2.2, ii)}}{\leq} \sup_{(a,b) \in R} h^*(M_a, N_b).$$

□

Given a set I and $n \in \mathbb{N}$, by $\Lambda_n(I)$ we designate $I^{\{1,\ldots,n\}}$, so the elements of $\Lambda_n(I)$ are finite words with n letters, having the form $\omega = \omega_1\omega_2 \ldots \omega_n$, where $\omega_k \in I$ for every $k \in \{1, \ldots, n\}$. The integer n is called the length of ω, and it is denoted by $|\omega|$.

By λ we designate the empty word, and we put $\Lambda_0(I) = \{\lambda\}$.

Given $m \in \{1, \ldots, n\}$ and $\omega = \omega_1\omega_2 \ldots \omega_n \in \Lambda_n(I)$, by $[\omega]_m$ we mean the element of $\Lambda_m(I)$ described by $\omega_1\omega_2 \ldots \omega_m$. Additionally, $[\omega]_0 = \lambda$.

By $\Lambda^*(I)$ we mean $\bigcup_{n \in \mathbb{N} \cup \{0\}} \Lambda_n(I)$.

By the concatenation of the words $\alpha = \alpha_1\alpha_2 \ldots \alpha_n \in \Lambda_n(I)$ and $\beta = \beta_1\beta_2 \ldots \beta_p \in \Lambda_p(I)$, we mean the element $\alpha\beta \in \Lambda_{n+p}(I)$ described by $\alpha_1 \ldots \alpha_n\beta_1 \ldots \beta_p$.

Given $f_i \colon X \to X$, $i \in I$, and $\omega = \omega_1\omega_2\ldots\omega_n \in \Lambda_n(I)$, $n \in \mathbb{N}$, we adopt the following notation:

$$f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n} =: f_\omega, \text{ and, additionally, } Id_X =: f_\lambda.$$

In particular if I has just one element, let us call it i, then we have $f_{\underbrace{i\ldots i}_{n \text{ times}}} = f_i^{[n]}$ for every $n \in \mathbb{N}$.

Definition 2.5 A possibly infinite iterated function system (for short IIFS) is a pair $((X,d),(f_i)_{i\in I}) \overset{not}{=} \mathcal{S}$, where (X,d) is a complete metric space and $f_i \colon X \to X$, $i \in I$, are such that:

(a) f_i is continuous for every $i \in I$;
(b) the family $(f_i)_{i\in I}$ is bounded, i.e., for every $B \in P_b(X)$, we have

$$\bigcup_{i\in I} f_i(B) \in P_b(X).$$

The function $F_{\mathcal{S}} \colon P_{b,cl}(X) \to P_{b,cl}(X)$, given by

$$F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)},$$

for every $B \in P_{b,cl}(X)$, is called the fractal operator associated with $\mathcal{S}$.

Proposition 2.6 *For each IIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I})$*, we have*

$$F_{\mathcal{S}}^{[n]}(B) = \overline{\bigcup_{\alpha\in\Lambda_n(I)} f_\alpha(B)},$$

for every $n \in \mathbb{N}$ *and every* $B \in P_{b,cl}(X)$.

Proof See Lemma 2.6 from [42]. □

Corollary 2.7 *For each IIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I})$*, we have*

$$F_{\mathcal{S}}^{[n]}(B) = \overline{\bigcup_{x\in B} F_{\mathcal{S}}^{[n]}(\{x\})},$$

for every $B \in P_{b,cl}(X)$.

Proof On the one hand, we have

$$F_{\mathcal{S}}^{[n]}(B) \overset{\text{Proposition 2.6}}{=} \overline{\bigcup_{x\in B}\bigcup_{\alpha\in\Lambda_n(I)} f_\alpha(x)} \subseteq \overline{\bigcup_{x\in B}\overline{\bigcup_{\alpha\in\Lambda_n(I)} f_\alpha(x)}} \overset{\text{Proposition 2.6}}{=} \overline{\bigcup_{x\in B} F_{\mathcal{S}}^{[n]}(\{x\})}.$$

On the other hand, we have

$$\overline{\bigcup_{x\in B} F_{\mathcal{S}}^{[n]}(\{x\})} \subseteq F_{\mathcal{S}}^{[n]}(B).$$

Indeed, as

$$F_{\mathcal{S}}^{[n]}(\{x\}) \overset{\text{Proposition 2.6}}{\subseteq} F_{\mathcal{S}}^{[n]}(B),$$

for all $x \in B$, we get

$$\bigcup_{x \in B} F_{\mathcal{S}}^{[n]}(\{x\}) \subseteq F_{\mathcal{S}}^{[n]}(B),$$

so, by the closedness of $F_{\mathcal{S}}^{[n]}(B)$, we also have

$$\overline{\bigcup_{x \in B} F_{\mathcal{S}}^{[n]}(\{x\})} \subseteq F_{\mathcal{S}}^{[n]}(B).$$

□

Given $x \in X$, $B \subseteq X$ and a family of functions $\mathcal{F} = (f_i)_{i \in I}$, where $f_i \colon X \to X$, we shall use the notation:

$$\bigcup_{n \in \mathbb{N} \cup \{0\}} \overline{\bigcup_{\alpha \in \Lambda_n(I)} f_\alpha(B)} =: \mathcal{O}_{\mathcal{F}}(B)$$

and

$$\mathcal{O}_{\mathcal{F}}(\{x\}) =: \mathcal{O}_{\mathcal{F}}(x).$$

In particular, given an IIFS $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ and $B \in P_{b,cl}(X)$, we shall use the notation

$$\mathcal{O}_{(f_i)_{i \in I}}(B) \overset{\text{Proposition 2.6}}{=} \bigcup_{n \in \mathbb{N} \cup \{0\}} F_{\mathcal{S}}^{[n]}(B) =: \mathcal{O}_{\mathcal{S}}(B).$$

Note that $f_i(O_{\mathcal{S}}(B)) \subseteq O_{\mathcal{S}}(B)$, consequently $f_i(\overline{O_{\mathcal{S}}(B)}) \subseteq \overline{O_{\mathcal{S}}(B)}$ for every $i \in I$ and $B \in P_{b,cl}(X)$.

Definition 2.8 A mixed possibly infinite iterated function system (briefly mIIFS) is an IIFS $\mathcal{S} =((X, d), (f_i)_{i \in I \cup J})$, where I and J are disjoint sets, such that:

(a) For every $i \in I \cup J$, we have

$$lip(f_i) \leq 1.$$

(b) There exists $a \in [0, 1)$ having the following properties:

  (bi) For every $i \in I$, we have

$$lip(f_i) \leq a.$$

  (bii) For every $x \in X$ and every $i \in J$, we have

$$lip(f_{i|\overline{\mathcal{O}_{(f_i)_{i \in J}}(x)}}) \leq a.$$

Remark 2.9 The terminology used in the above definition is based on the following two facts:

(i) $((X, d), (f_i)_{i \in I}) := \mathcal{S}_I$ is an IIFS such that there exists $a \in [0, 1)$ having the property that

$$lip(f_i) \leq a,$$

for every $i \in I$.

(ii) $((X, d), (f_i)_{i \in J}) := \mathcal{S}_J$ is an oIIFS in the sense of [29], satisfying the additional condition that the maps are nonexpansive.

In the framework of the above definition, for $B \in P_{b,cl}(X)$ and $\alpha = \alpha_1 \dots \alpha_p \in \Lambda^*(I \cup J)$, we shall use the following notation:

$$
\begin{array}{ll}
F_{\mathcal{S}_I} =: F_I & F_{\mathcal{S}_J} =: F_J \\
\mathcal{O}_{\mathcal{S}_I}(B) =: \mathcal{O}_I(B) & \mathcal{O}_{\mathcal{S}_J}(B) =: \mathcal{O}_J(B) \\
card(\{l \in \{1, \dots, p\} \mid \alpha_l \in I\}) =: n_I(\alpha) & card(\{l \in \{1, \dots, p\} \mid \alpha_l \in J\}) =: n_J(\alpha).
\end{array}
$$

Definition 2.10 By a J block of $\alpha = \alpha_1 \dots \alpha_n \in \Lambda^*(I \cup J)$ we mean an element $\alpha_k \alpha_{k+1} \dots \alpha_l \in \Lambda^*(J)$ such that $\{k, k+1, \dots, l\} \subseteq \{1, 2, \dots, n\}$.

Definition 2.11 For $\alpha \in \Lambda^*(I \cup J)$, $\beta \in \Lambda^*(I \cup J) \backslash \Lambda^*(I)$ and $p = \max\{|\gamma| \mid \gamma$ is a J block of $\beta\} - 1$ we say that $\alpha \overset{p}{\sim} \beta$ if α is obtained from β by eliminating the last letter of a J block of β having length $p + 1$.

We provide an example illustrating the above definition.

Example 2.12 Let $\alpha_1, \alpha_2, \alpha_3, \beta \in \Lambda^*(I \cup J)$ given by

$$
\begin{aligned}
\alpha_1 &= i_1 i_2 j_1 j_2 i_3 j_4 j_5 j_6 j_7 i_4 j_8 j_9 j_{10} j_{11} i_5, \\
\alpha_2 &= i_1 i_2 j_1 j_2 j_3 i_3 j_4 j_5 j_6 i_4 j_8 j_9 j_{10} j_{11} i_5, \\
\alpha_3 &= i_1 i_2 j_1 j_2 j_3 i_3 j_4 j_5 j_6 j_7 i_4 j_8 j_9 j_{10} i_5
\end{aligned}
$$

and

$$
\beta = i_1 i_2 j_1 j_2 j_3 i_3 j_4 j_5 j_6 j_7 i_4 j_8 j_9 j_{10} j_{11} i_5,
$$

where $i_1, \dots, i_5 \in I$ and $j_1, \dots, j_{11} \in J$, I and J being disjoint sets.

Notice that $\alpha_2 \overset{3}{\sim} \beta$ and $\alpha_3 \overset{3}{\sim} \beta$, but $\alpha_1 \overset{2}{\sim} \beta$ is not true since α_1 is obtained from β by eliminating the last letter of the J block $j_1 j_2 j_3$, but β contains a J block having length 4.

3 The main result

We start this section with five technical lemmas and a proposition which will be used in the proof of our main result, namely Theorem 3.7, stating that the fractal operator associated with a mixed possibly infinite iterated function system is weakly Picard.

For an mIIFS $\mathcal{S} = ((X, d), (f_i)_{i \in I \cup J})$ and $n \in \mathbb{N}$, $n \geq 4$, let us consider

$$
\begin{aligned}
A &= \left\{\alpha \in \Lambda_n(I \cup J) | n_I(\alpha) \geq [\sqrt{n}]\right\}, \\
B &= \left\{\alpha i \in \Lambda_{n+1}(I \cup J) | \alpha \in \Lambda_n(I \cup J),\ n_I(\alpha) \geq [\sqrt{n}] \text{ and } i \in I \cup J\right\}, \\
\tilde{A} &= \left\{ \alpha = \alpha_1 \dots \alpha_n \in \Lambda_n(I \cup J) \,\middle|\, \begin{array}{c} \text{there exist } p \in \mathbb{N} \cup \{0\} \text{ and } s \in \mathbb{N} \text{ with } s \geq \left[\frac{n - [\sqrt{n}]}{[\sqrt{n}] + 1}\right] - 1 \\ \text{such that } \alpha_{p+1} \dots \alpha_{p+s} \in \Lambda^*(J) \end{array} \right\}
\end{aligned}
$$

and

$$
\tilde{B} = \left\{ \alpha = \alpha_1 \dots \alpha_{n+1} \in \Lambda_{n+1}(I \cup J) \,\middle|\, \begin{array}{c} \text{there exist } p \in \mathbb{N} \cup \{0\} \text{ and } s \in \mathbb{N} \text{ with } s \geq \left[\frac{n - [\sqrt{n}]}{[\sqrt{n}] + 1}\right] - 1 \\ \text{such that } \alpha_{p+1} \dots \alpha_{p+s} \alpha_{p+s+1} \in \Lambda^*(J) \end{array} \right\}.
$$

To keep the notations simple we use A, B, $\tilde{A}$ and $\tilde{B}$ even though these sets depend on n.

Lemma 3.1 *In the above framework, we have*

$$
\Lambda_n(I \cup J) = A \cup \tilde{A}.
$$

Proof If $\alpha \in \Lambda_n(I \cup J)\backslash A$, then $n_I(\alpha) < [\sqrt{n}]$.

Since $n_I(\alpha) + n_J(\alpha) = n$, we have

$$n_J(\alpha) > n - [\sqrt{n}]. \tag{1}$$

There exist $k \in \mathbb{N}$, $\beta_1, \beta_{k+1} \in \Lambda^*(I)$, $\beta_2, \ldots, \beta_k \in \Lambda^*(I)\backslash\{\lambda\}$ and $\gamma_1, \ldots, \gamma_k \in \Lambda^*(J)\backslash\{\lambda\}$ such that

$$\alpha = \beta_1\gamma_1\beta_2\gamma_2 \ldots \beta_k\gamma_k\beta_{k+1}.$$

We have $k - 1 \leq n_I(\alpha) < [\sqrt{n}]$, so

$$k < [\sqrt{n}] + 1. \tag{2}$$

In view of the generalized pigeonhole principle, there exists $t \in \{1, \ldots, k\}$ such that

$$|\gamma_t| \geq \left\lceil \frac{n_J(\alpha)}{k} \right\rceil \overset{(1)\&(2)}{\geq} \left\lceil \frac{n - [\sqrt{n}]}{[\sqrt{n}] + 1} \right\rceil > \left\lceil \frac{n - [\sqrt{n}]}{[\sqrt{n}] + 1} \right\rceil - 1.$$

Hence, $\alpha \in \tilde{A}$. □

Similarly, we deduce the following result:

Lemma 3.2 *In the above framework, we have*

$$\Lambda_{n+1}(I \cup J) = B \cup \tilde{B}.$$

Proposition 3.3 *In the above framework, for each $x \in X$, we have*

(i)

$$h^*\left(\bigcup_{\alpha \in A}\{f_\alpha(x)\}, \bigcup_{\alpha \in A, i \in I \cup J}\{f_{\alpha i}(x)\}\right) \leq \sup_{\alpha \in A, i \in I \cup J} d(f_\alpha(x), f_{\alpha i}(x)).$$

(ii)

$$h^*\left(\bigcup_{\alpha \in \tilde{A}}\{f_\alpha(x)\}, \bigcup_{\beta \in \tilde{B}}\{f_\beta(x)\}\right) \leq \sup_{\substack{\alpha \in \tilde{A}, \beta \in \tilde{B} \\ \text{such that } \alpha \overset{\ell}{\sim} \beta \text{ for some } \ell \geq \left\lceil \frac{n-[\sqrt{n}]}{[\sqrt{n}]+1} \right\rceil - 1}} d(f_\alpha(x), f_\beta(x)).$$

Proof (i) It follows, via Lemma 2.4, taking A, B, $R = \{(\alpha, \alpha i) | \alpha \in A \text{ and } i \in I \cup J\}$, $(M_a)_{a \in A} = (\{f_\alpha(x)\})_{\alpha \in A}$ and $(N_b)_{b \in B} = (\{f_\beta(x)\})_{\beta \in B}$.

(ii) It results from Lemma 2.4 considering:

$\tilde{A}$, $\tilde{B}$, $R = \left\{(\alpha, \beta) | \alpha \in \tilde{A},\ \beta \in \tilde{B} \text{ are such that } \alpha \overset{\ell}{\sim} \beta \text{ for some } \ell \geq \left\lceil \frac{n-[\sqrt{n}]}{[\sqrt{n}]+1} \right\rceil - 1\right\}$, $(M_a)_{a \in \tilde{A}} = (\{f_\alpha(x)\})_{\alpha \in \tilde{A}}$ and $(N_b)_{b \in \tilde{B}} = (\{f_\beta(x)\})_{\beta \in \tilde{B}}$.

Indeed, on the one hand, for each $\alpha \in \tilde{A}$ there exists $\beta \in \tilde{B}$ such that $\alpha \overset{\ell}{\sim} \beta$ for some $\ell \geq \left\lceil \frac{n-[\sqrt{n}]}{[\sqrt{n}]+1} \right\rceil - 1$. Specifically, β is obtained from α by inserting an arbitrarily chosen letter from J at the end of a J block of α having maximal length (which is at least $\left\lceil \frac{n-[\sqrt{n}]}{[\sqrt{n}]+1} \right\rceil - 1$).

On the other hand, for each $\beta \in \tilde{B}$, there exists $\alpha \in \tilde{A}$ such that $\alpha \overset{\ell}{\sim} \beta$ for some $\ell \geq \left\lceil \frac{n-[\sqrt{n}]}{[\sqrt{n}]+1} \right\rceil - 1$. Namely, α is obtained from β by eliminating the last letter of a J block of β having maximal length (which is at least $\left\lceil \frac{n-[\sqrt{n}]}{[\sqrt{n}]+1} \right\rceil$).

So the hypotheses of Lemma 2.4 are satisfied. □

Lemma 3.4 *For each mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ *and* $\alpha \in \Lambda^*(I\cup J)$*, we have*

$$lip(f_\alpha) \le a^{n_I(\alpha)}.$$

Proof For $\alpha = \alpha_1\dots\alpha_p \in \Lambda^*(I\cup J)$, we have

$$lip(f_\alpha) = lip(f_{\alpha_1\dots\alpha_p}) \le lip(f_{\alpha_1})\cdot\dots\cdot lip(f_{\alpha_p}) \overset{\text{Remark 2.9.}}{\le} \underbrace{a\cdot\dots\cdot a}_{n_I(\alpha)\text{ times}} = a^{n_I(\alpha)}.$$

□

Lemma 3.5 *For each mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$*, the pair*

$$\mathfrak{S} = ((X,d),(f_{\omega i\theta})_{i\in I,\omega,\theta\in\Lambda^*(J)})$$

is an IIFS.

Proof Indeed, on the one hand, we have

$$lip(f_{\omega i\theta}) \overset{\text{Lemma 3.4}}{\le} a, \tag{3}$$

for all $i \in I$ and $\omega,\theta \in \Lambda^*(J)$.

On the other hand, first let us note that, according to Lemma 3.36 from [29], we have

$$\mathcal{O}_J(B) \in P_b(X), \tag{4}$$

for all $B \in P_b(X)$. Moreover,

$$\begin{aligned} f_{\omega i\theta}(B) &= (f_\omega\circ f_i)(f_\theta(B)) \subseteq (f_\omega\circ f_i)(\overline{\mathcal{O}_J(B)}) \subseteq \\ &\subseteq f_\omega(F_I(\overline{\mathcal{O}_J(B)})) \subseteq \overline{\mathcal{O}_J(F_I(\overline{\mathcal{O}_J(B)}))}, \end{aligned}$$

for all $B \in P_b(X)$ and all $i \in I$, $\omega,\theta \in \Lambda^*(J)$, so

$$F_{\mathfrak{S}}(B) \subseteq \overline{\mathcal{O}_J(F_I(\overline{\mathcal{O}_J(B)}))} \overset{(4)}{\in} P_b(X), \tag{5}$$

for all $B \in P_b(X)$.

Taking into account (3) and (5), the proof is complete. □

Lemma 3.6 *For each mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$*,* $x \in X$*,* $\alpha \in \Lambda^*(I\cup J)$*,* $\beta \in \Lambda^*(I\cup J)\setminus\Lambda^*(I)$ *and* $p \in \mathbb{N}$ *such that* $\alpha \overset{p}{\sim} \beta$*, we have*

$$d(f_\alpha(x), f_\beta(x)) \le a^p \max\{diam(\mathcal{O}_{\mathfrak{S}}(x)), diam(\mathcal{O}_J(x))\}.$$

Proof For $\alpha \in \Lambda^*(I\cup J)$, $\beta \in \Lambda^*(I\cup J)\setminus\Lambda^*(I)$ and $p \in \mathbb{N}$ such that $\alpha \overset{p}{\sim} \beta$, for some $m \in \mathbb{N}$, we have

$$\beta = \beta_1\dots\beta_m\beta_{m+1}\dots\beta_{m+p}\beta_{m+p+1}\beta_{m+p+2}\dots\beta_{|\beta|}$$

and

$$\alpha = \beta_1\dots\beta_m\beta_{m+1}\dots\beta_{m+p}\beta_{m+p+2}\dots\beta_{|\beta|}$$

with

$$\beta_{m+1}\dots\beta_{m+p}\beta_{m+p+1} \in \Lambda^*(J). \tag{6}$$

We have

$$\begin{aligned} d(f_\alpha(x), f_\beta(x)) &= d(f_{\beta_1\ldots\beta_m\beta_{m+1}\ldots\beta_{m+p}\beta_{m+p+2}\cdots\beta_{|\beta|}}(x), f_{\beta_1\ldots\beta_m\beta_{m+1}\ldots\beta_{m+p}\beta_{m+p+1}\beta_{m+p+2}\cdots\beta_{|\beta|}}(x)) \\ &\overset{lip(f_{\beta_1\ldots\beta_m})\leq 1}{\leq} d\left(f_{\beta_{m+1}\ldots\beta_{m+p}}(f_{\beta_{m+p+2}\cdots\beta_{|\beta|}}(x)), f_{\beta_{m+1}\ldots\beta_{m+p}}(f_{\beta_{m+p+1}\beta_{m+p+2}\cdots\beta_{|\beta|}}(x))\right) \\ &\overset{\text{Definition 2.8, } b\ ii)}{\leq} a^p d(f_{\beta_{m+p+2}\cdots\beta_{|\beta|}}(x), f_{\beta_{m+p+1}}(f_{\beta_{m+p+2}\cdots\beta_{|\beta|}}(x))). \end{aligned} \tag{7}$$

We divide our discussion into two cases:

(i) $\beta_{m+p+2}\ldots\beta_{|\beta|} = \lambda$
(ii) $\beta_{m+p+2}\ldots\beta_{|\beta|} \neq \lambda$.

In the first case, via (6) and (7), we get

$$d(f_\alpha(x), f_\beta(x)) \leq a^p diam(\mathcal{O}_J(x)). \tag{8}$$

In the second case, as the maximal length of a J block of β is $p+1$, $\beta_{m+p+2} \in I$, so

$$f_{\beta_{m+p+2}\cdots\beta_{|\beta|}}(x), f_{\beta_{m+p+1}}(f_{\beta_{m+p+2}\cdots\beta_{|\beta|}}(x)) \in \mathcal{O}_{\mathfrak{S}}(x)$$

and, via (7), we infer that

$$d(f_\alpha(x), f_\beta(x)) \leq a^p diam(\mathcal{O}_{\mathfrak{S}}(x)). \tag{9}$$

Based on (8) and (9), the proof is complete. □

Theorem 3.7 *$F_\mathcal{S}$ is weakly Picard for each mIIFS $\mathcal{S} =((X,d),(f_i)_{i\in I\cup J})$. More precisely, for each $C \in P_{b,cl}(X)$ there exists $A_C \in P_{b,cl}(X)$ such that*

$$\lim_{n\to\infty} F_\mathcal{S}^{[n]}(C) = A_C = F_\mathcal{S}(A_C).$$

Proof For $x \in X$ and $C \in P_{b,cl}(X)$, we shall use the following notation:

$$\max\{diam(\{x\} \cup F_\mathcal{S}(\{x\}), diam(\mathcal{O}_{\mathfrak{S}}(x)), diam(\mathcal{O}_J(x))\} := N_x$$

and

$$\sup_{x\in C} N_x := N_C \leq \max\{diam(C \cup F_\mathcal{S}(C)), diam(\mathcal{O}_{\mathfrak{S}}(C)), diam(\mathcal{O}_J(C))\} \overset{\text{Lemma 3.5}}{<} \infty.$$

Claim 1

$$h(F_\mathcal{S}^{[n]}(\{x\}), F_\mathcal{S}^{[n+1]}(\{x\})) \leq N_x a^{min\left\{[\sqrt{n}], \left[\frac{n+1}{[\sqrt{n}]+1}\right]-2\right\}},$$

for every $x \in X$ and $n \in \mathbb{N}$.

Justification of Claim 1. We have

$$\begin{aligned} h(F_\mathcal{S}^{[n]}(\{x\}), F_\mathcal{S}^{[n+1]}(\{x\})) &\overset{\substack{\text{Proposition 2.2 \&} \\ \text{Proposition 2.6}}}{=} h^*\left(\bigcup_{\alpha\in\Lambda_n(I\cup J)} \{f_\alpha(x)\}, \bigcup_{\beta\in\Lambda_{n+1}(I\cup J)} \{f_\beta(x)\}\right) \\ &\overset{\substack{\text{Lemma 3.1 \&} \\ \text{Lemma 3.2}}}{=} h^*\left(\bigcup_{\alpha\in A}\{f_\alpha(x)\} \cup \bigcup_{\alpha\in\tilde{A}}\{f_\alpha(x)\}, \bigcup_{\beta\in B}\{f_\beta(x)\} \cup \bigcup_{\beta\in\tilde{B}}\{f_\beta(x)\}\right) \\ &\overset{\text{Proposition 2.2}}{\leq} \max\left\{h^*\left(\bigcup_{\alpha\in A}\{f_\alpha(x)\}, \bigcup_{\beta\in B}\{f_\beta(x)\}\right), h^*\left(\bigcup_{\alpha\in\tilde{A}}\{f_\alpha(x)\}, \bigcup_{\beta\in\tilde{B}}\{f_\beta(x)\}\right)\right\} \end{aligned}$$

$$\overset{\text{Proposition 3.3}}{\leq} \max \left\{ \sup_{\alpha \in A, i \in I \cup J} d(f_\alpha(x), f_{\alpha i}(x)), \sup_{\substack{\alpha \in \tilde{A}, \beta \in \tilde{B} \\ \text{such that } \alpha \overset{\ell}{\sim} \beta \text{ for some } \ell \geq \left[\frac{n-[\sqrt{n}]}{[\sqrt{n}]+1}\right]-1}} d(f_\alpha(x), f_\beta(x)) \right\}$$

$$\overset{\substack{\text{Lemma 3.4 \&} \\ \text{Lemma 3.6}}}{\leq} \max \left\{ a^{[\sqrt{n}]} diam(\{x\} \cup F_{\mathcal{S}}(\{x\})), a^{\left[\frac{n-[\sqrt{n}]}{[\sqrt{n}]+1}\right]-1} \max\{diam(\mathcal{O}_{\mathfrak{S}}(x)), diam(\mathcal{O}_J(x))\} \right\}$$

$$\leq N_x a^{\min\left\{[\sqrt{n}], \left[\frac{n-[\sqrt{n}]}{[\sqrt{n}]+1}\right]-1\right\}} = N_x a^{\min\left\{[\sqrt{n}], \left[\frac{n+1}{[\sqrt{n}]+1}\right]-2\right\}},$$

for every $x \in X$ and $n \in \mathbb{N}$. Hence, the justification of the Claim in complete.

Claim 2 There exists $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that

$$\lim_{n \to \infty} (x_n - \sqrt{n}) \in \mathbb{R}$$

and

$$h(F_{\mathcal{S}}^{[n]}(\{x\}), F_{\mathcal{S}}^{[n+1]}(\{x\})) \leq N_x a^{x_n},$$

for every $x \in X$ and $n \in \mathbb{N}$.

Justification of Claim 2. First let us note that, since $\frac{n}{[\sqrt{n}]+1} < [\sqrt{n}] + 1$, we have $\frac{n+1}{[\sqrt{n}]+1} < [\sqrt{n}] + 1 + \frac{1}{[\sqrt{n}]+1}$, so

$$\left[\frac{n+1}{[\sqrt{n}]+1}\right] \leq [\sqrt{n}] + 2, \tag{10}$$

for every $n \in \mathbb{N}$.

Thus,

$$a^{\min\left\{[\sqrt{n}], \left[\frac{n+1}{[\sqrt{n}]+1}\right]-2\right\}} \overset{(10)}{=} a^{\left[\frac{n+1}{[\sqrt{n}]+1}\right]-2} < a^{\frac{n+1}{\sqrt{n}+1}-3},$$

for every $n \in \mathbb{N}$ and, via Claim 1, the justification of Claim 2 is complete by choosing $x_n = \frac{n+1}{\sqrt{n}+1} - 3$ for every $n \in \mathbb{N}$.

Then

$$h(F_{\mathcal{S}}^{[n]}(C), F_{\mathcal{S}}^{[n+1]}(C)) \overset{\text{Proposition 2.2 \& Corollary 2.7}}{\leq} \sup_{x \in C} h(F_{\mathcal{S}}^{[n]}(\{x\}), F_{\mathcal{S}}^{[n+1]}(\{x\}))$$

$$\overset{\text{Claim 2}}{\leq} \sup_{x \in C} N_x a^{x_n} = a^{x_n} N_C, \tag{11}$$

for every $C \in P_{b,cl}(X)$ and $n \in \mathbb{N}$. So, in view of (11), $(F_{\mathcal{S}}^{[n]}(C))_{n \in \mathbb{N}}$ is Cauchy and consequently there exists $A_C \in P_{b,cl}(X)$ such that

$$\lim_{n \to \infty} F_{\mathcal{S}}^{[n]}(C) = A_C.$$

As $F_{\mathcal{S}}$ is continuous (since $lip(F_{\mathcal{S}}) \leq 1$), we infer that

$$F_{\mathcal{S}}(A_C) = A_C.$$

□

Remark 3.8 In the framework of the above theorem, we have:

(i)

$$h(F_{\mathcal{S}}^{[n]}(C), A_C) \leq N_C \sum_{k=n}^{\infty} a^{x_k},$$

for each $n \in \mathbb{N}$ and $C \in P_{b,cl}(X)$.
Indeed, for each $C \in P_{b,cl}(X)$ and $n, p \in \mathbb{N}$, we have

$$\begin{aligned} h(F_{\mathcal{S}}^{[n]}(C), A_C) &\leq \sum_{k=n}^{n+p} h(F_{\mathcal{S}}^{[k]}(C), F_{\mathcal{S}}^{[k+1]}(C)) + h(F_{\mathcal{S}}^{[n+p+1]}(C), A_C) \\ &\overset{(11)}{\leq} N_C \sum_{k=n}^{n+p} a^{x_k} + h(F_{\mathcal{S}}^{[n+p+1]}(C), A_C), \end{aligned}$$

so, passing to limit as p goes to ∞ in the above inequality, we obtain

$$h(F_{\mathcal{S}}^{[n]}(C), A_C) \leq N_C \sum_{k=n}^{\infty} a^{x_k}.$$

(ii) If $I \cup J$ is finite and $K \in P_{cp}(X)$, then $A_K \in P_{cp}(X)$.
Indeed, $F_{\mathcal{S}}^{[n]}(K) \in P_{cp}(X)$ for all $n \in \mathbb{N}$, so, based on Proposition 2.7 from [26], we conclude that $\lim_{n\to\infty} F_{\mathcal{S}}^{[n]}(K) = A_K \in P_{cp}(X)$.

4 Examples

Example 4.1 (Fig. 1) Let us consider $\mathcal{S} = ((\mathbb{R}^2, \|\cdot\|_2), (f_i)_{i\in I\cup J})$, where $I = \{1, 2, 3\}$, $J = \{4\}$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$ and $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$\begin{aligned} &f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \; f_2(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right), \\ &f_3(x, y) = \left(\frac{2x+1}{4}, \frac{2y+\sqrt{3}}{4}\right) \text{ and } f_4(x, y) = (x, \alpha y + \beta), \end{aligned}$$

for every $(x, y) \in \mathbb{R}^2$.

$\mathcal{S}$ is an mIIFS, but it is not an oIIFS.

Indeed, observe that $lip(f_1) = lip(f_2) = lip(f_3) = \frac{1}{2}$ and

$$\overline{\mathcal{O}_J((u, v))} = \left\{ \left(u, \alpha^n v + \beta \frac{1-\alpha^n}{1-\alpha}\right) \middle| \, n \in \mathbb{N} \cup \{0\} \right\} \cup \left\{ \left(u, \frac{\beta}{1-\alpha}\right) \right\},$$

for all $(u, v) \in \mathbb{R}^2$.

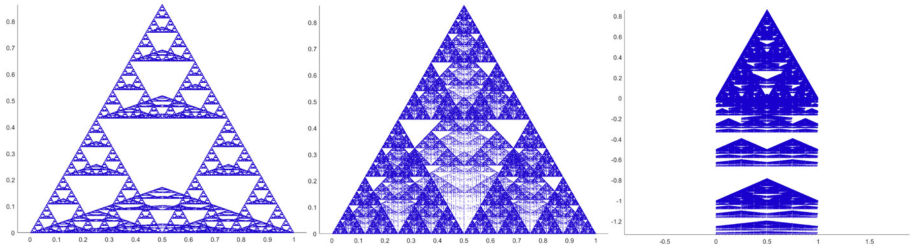
Hence, we have

$$||f_4(z) - f_4(t)||_2 = \alpha ||z - t||_2,$$

for all $z, t \in \overline{\mathcal{O}_J(x)}$ and $x \in \mathbb{R}^2$.

Therefore,

$$lip\left(f_4|_{\overline{\mathcal{O}_J(x)}}\right) = \alpha < 1,$$



$F_{\mathcal{S}}^{[10]}(\{(1,0)\})$
$\alpha=\frac{1}{5}$ and $\beta=0$

$F_{\mathcal{S}}^{[10]}(\{(1,0)\})$
$\alpha=\frac{3}{5}$ and $\beta=0$

$F_{\mathcal{S}}^{[10]}(\{(0,0)\})$
$\alpha=\frac{1}{4}$ and $\beta=-1$

Fig. 1 Graphical representations for Example 4.1

for all $x \in \mathbb{R}^2$. So $\mathcal{S}$ is an mIIFS with $a = \max\left\{\alpha, \frac{1}{2}\right\}$.

Additionally, for any arbitrary $x \in \mathbb{R}^2$, note that $f_1(x), f_2(x) \in \overline{\mathcal{O}_{\mathcal{S}}(x)}$, and we have

$$||f_4(f_1(x)) - f_4(f_2(x))||_2 = ||f_1(x) - f_2(x)||_2,$$

so

$$lip\left(f_4|_{\overline{\mathcal{O}_{\mathcal{S}}(x)}}\right) \geq 1.$$

Therefore, $\mathcal{S}$ is not an oIIFS.

Example 4.2 Let us consider $\mathcal{S} = ((\mathbb{R}^2, \|.\|_2), (f_i)_{i\in I\cup J})$, where $I = \{1, 2\}$, $J = \{3\}$, $a, r \in (0, 1)$, $\alpha \in \mathbb{R}$ and $f_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$f_1(x, y) = r\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix},$$

$$f_2(x, y) = r\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}$$

and

$$f_3(x, y) = (x, ay),$$

for every $(x, y) \in \mathbb{R}^2$.

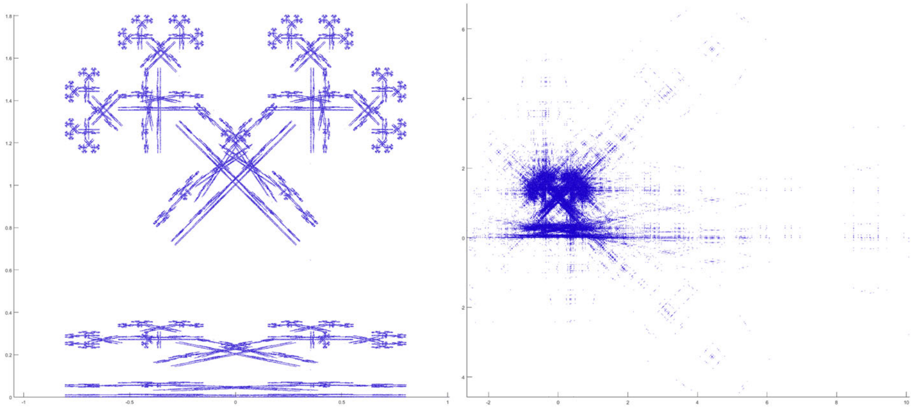
We stress upon the fact that the graphical representations from Fig. 2 are generated starting from different sets, namely $\{(1, 0)\}$ and $\{(100, 0)\}$.

Example 4.3 (Fig. 3) Let us consider $\mathcal{S} = ((\mathbb{R}^2, \|.\|_2), (f_i)_{i\in I\cup J})$, where $I = \{1\}$, $J = \{2, 3\}$, $a, r \in (0, 1)$, $\alpha \in \mathbb{R}$ and $f_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$f_1(x, y) = r\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},\ f_2(x, y) = (x, ay + 1 - a)$$
$$\text{and } f_3(x, y) = (x, ay - 1 + a),$$

for every $(x, y) \in \mathbb{R}^2$.

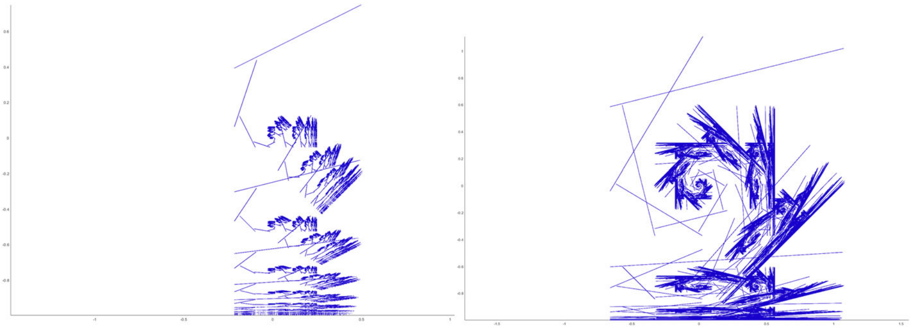
Finally, we present an example of an mIIFS such that the number of its constitutive functions is infinite.



$F_{\mathcal{S}}^{[20]}(\{(1,0)\})$
$r=\frac{1}{2}$, $a=\frac{1}{5}$ and $\alpha=\frac{\pi}{4}$

$F_{\mathcal{S}}^{[20]}(\{(100,0)\})$
$r=\frac{1}{2}$, $a=\frac{1}{5}$ and $\alpha=\frac{\pi}{4}$

Fig. 2 Graphical representations for Example 4.2



$F_{\mathcal{S}}^{[20]}(\{(0,0)\})$
$r=a=\frac{1}{2}$ and $\alpha=\frac{\pi}{4}$

$F_{\mathcal{S}}^{[20]}(\{(0,0)\})$
$r=\frac{3}{4}$, $a=\frac{1}{4}$ and $\alpha=\frac{\pi}{4}$

Fig. 3 Graphical representations for Example 4.3

Example 4.4 Let (X, d_X) and (Y, d_Y) be complete metric spaces, $A \in P_{b,cl}(X \times Y)$, $I \subseteq A$, where X, A and I are infinite, $X \times Y$ is endowed with the maximum metric d_{max} and $\overline{I} = A$. Consider an infinite set J, such that $I \cap J = \emptyset$, and a family of continuous functions $g_j\colon Y \to X$, $j \in J$, so that

$$\bigcup_{j \in J} g_j(Y) \in P_b(X).$$

Then $\mathcal{S} = ((X \times Y, d_{max}), (f_i)_{i \in I \cup J})$ is an mIIFS, where, for all $(x, y) \in X \times Y$,

$$f_i(x, y) = i,$$

if $i \in I$ and

$$f_i(x, y) = (g_i(y), y),$$

if $i \in J$.

Note that

$$A_B = A \cup \overline{\{(g_j(y), y) | j \in J \text{ and } y \in Y \text{ are such that there exists } x \in X \text{ with } (x, y) \in A \cup B\}},$$

for all $B \in P_{b,cl}(X \times Y)$.

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Data availability Not applicable.

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