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Enhancing Efficient Data Transmission in IBM WebSphere Using Relational Data eXchange (RDX) Mechanism and Tandem Queue

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Abstract: This study investigates tandem queues with two service nodes. The consumer needs to obtain services at the following two nodes in this system: the IBM online sphere for artificial intelligence (AI) and the complex RDX mechanism. Before the second essential service (SES), by utilizing AI to validate the data in IBM WebSphere and insight, the first essential service (FES) begins with the RDX mechanism. If there are fewer customers than "a" after a service at node 1 is finished, the server departs for a subsequent assignment. As soon as the vacation value reached the threshold, the service began. After the service concludes at node 1, it moves on to node 2. In this study, the supplemental variable technique is used to determine the probability-generating function (PGF) at any given time. A numerical solution also yields certain features of the queueing system.

Keywords: data transmission; RDX mechanism; AI; WebSphere; performance analysis

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1. Introduction

In a tandem network, several classes of jobs or service points exist, and in using the network, each new customer must transit each class of job in turn before leaving the system. An example of this is a car service center. In this case, a car that is introduced and must be thoroughly inspected at each service point could be an illustration of this process. Analogously, we can refer to a medical laboratory where, in order to obtain the results of a blood test, a person must go to several counters to obtain the results they need. They must first present themselves at reception, then at the doctor's office, and then at the blood collection counter [1]. For studying such cases, appropriate models have been proposed. For example, in order to analyze the flow of road traffic, a new M/G/c/c waiting model is presented in [2]. The method presented in the paper offers solutions for better infrastructure and traffic management and allows the understanding and evaluation of road traffic flow planning.

Energy harvesting in a wireless sensor node's queueing architecture is examined in [3], an investigation of a bulk queueing system undergoing a functioning breakdown is conducted in [4], and a practical illustration of such a system used in 4G/5G networks is provided [5]. Ref. [6] investigates the impact of the correlation between stage service times and total waiting time, including service time. The importance of the "buffer" space was underlined (interstage waiting). An approximation technique for examining open networks of queues simultaneously while taking blocking into account was put forth by [7]. The network presented in [8] is formed via M single-server queuing operations with exponentially distributed service times together with homogenous Poisson arrival processes (queues have Poisson distributions). The limitless number of service channels and the general distribution of service times set the queueing systems apart. The author was able to determine the time-dependent distribution of queue length by using the M/G/m tandem model. This method proved to be successful in assessing performance metrics in tandem queueing systems with finite buffers. Interestingly, it was discovered that starvation and blocking were important in these systems. According to the research conducted in [9], both the simulation and test results showed that the approach, although approximative in nature, had a high level of accuracy.

The status and first-come, first-served waiting time are analyzed in a paper presenting a three-stage queueing system. The system is distinguished by having little waiting room before the first stage and none between stages. Simplifying assumptions have been made in previous assessments of similar systems, such as the assumption that the initial stage is never empty. The study performed in [10] readily establishes the conditions sufficient for the existence of an equilibrium. An approximate method for examining tandem configurations—which are made up of a sequence of single-server finite queues—was put forth in [11]. Only departures from the last queue are permitted.

Reference [12] presents an entropy analysis of controlling the arrival and batches. In [13], the effects of reneging, server failure, and server vacation on the different phases of the batch arrivals are examined. The research in [14] examined the quality control policy using a Markovian model. The authors of [15] examined an open tandem queuing network. In order to reduce a queueing network to a two-node queueing, the researchers suggested a transformation technique. These limitations are computationally tractable and easy to compute. The study performed in [16] concentrated on examining a two-stage tandem queueing model in its steady state. There are two parallel queues at the first step. The authors use generating functions to obtain joint queue length distributions, one at each given moment and one immediately following service completion. In their theory of piecewise Markov processes, the authors applied the rate conservation principle.

This study also derives Laplace–Stieltjes transformations for certain sojourn time distributions. Additionally, the authors include actual data on average waiting durations and total sojourn times. A queueing analysis method is proposed in [17] to address the performance optimization issue. The authors construct an optimization principle and characterize macroscale pipeline processes. This technique is then used to analyze some applications, such as photo retrieval and intrusion detection.

In [18], a single-server retrial queue during vacations and working breaks was studied. If the required and sufficient circumstances are met, this system can be stabilized. By concentrating on obtaining performance boundaries without looking into the specifics of particular queues, [19] introduced a novel method for assessing queuing systems. Numerical experiments show the usefulness of these bounds as approximations in stochastic circumstances and examine their ramifications. By providing fresh methods for comprehending and assessing the performance of intricate queuing systems, this study advances the subject of queuing analysis.

A novel method for analyzing tandem queues with a finite buffer capacity that have interruptions to simulate bottleneck starvation has been investigated [20]. This study highlighted the connection between system service rates and arrival rates while illustrating the basic characteristics of such lineups. The authors used ideas from Friedman's reduction method to propose a novel way to analyze the performance of such queues. The marginal probabilities for each queue are calculated simultaneously in [21]. Two types of networks are analyzed as follows: one with a porous medium and another where customers can enter from a leading queue.

Rao et al. [22] looked at the M/M/1 vacation two-phase queueing model. A tandem queueing network involving clients with limited patience was investigated in [23]. They developed a model (called LDQBD) using a Markovian framework. Notably, the team investigated a certain kind of LDQBD process that shows a level-dependent characteristic, meaning that the overall desertion rate rises. In order to address this issue, the authors studied a different system that used a last-come, first-served methodology. So, they built a recursive link with the LDQBD method. Of interest, this operation has the potential to be precisely and efficiently completed using currently known numerical approaches. To provide a thorough examination of the system, the researchers also used integrated queueing and Markov chain decomposition (QMCD) techniques. Through the development and analysis of a tandem queuing model with feedback, [24] made contributions to the field of queuing networks. The methodology is employed in many other fields, and this study concentrated on assessing important performance indicators, including average waiting times and number of customers in the queue. The study offers important insights into system behavior and resource utilization by using difference–differential equations and probability-generating function approaches. Tandem queues with failures are investigated in [25]. Several researchers have conducted in-depth studies in this domain [26–31].

2. Materials and Methods

Customers are entering the system in large quantities at a rate of λ_1, as per the Poisson process. Clients requiring services at both minimum ($'a'$) and maximum ($'b'$) server capacity are separated into two terminal systems within a batch service process [26]. Customers proceed to node 2 and obtain the second service after joining the system at node 1 and waiting for service. If the queue length is greater than or equal to $'a'$, the server proceeds to the next batch of services when the node 1 service is completed; if it is less than or equal to $'a'$, the server takes a vacation. If the queue length exceeds or equals $'a'$ After a vacation, the server restarts the service.

Following node 1's service, clients proceed to node 2 for the subsequent service. This system is presented in a schematic form in Figure 1.

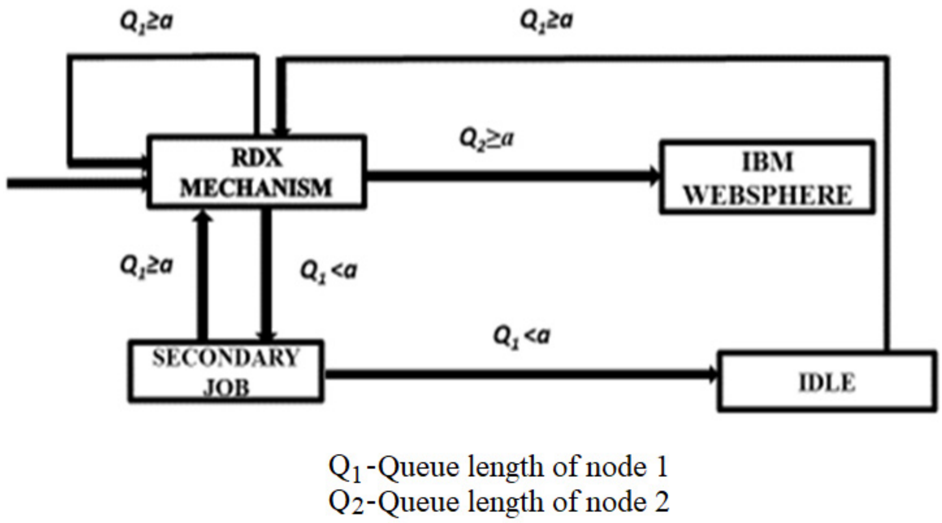


Figure 1. Sketch of the queueing model.

The purposeful queuing model is based on a real-world situation that occurs in the IBM WebSphere and RDX mechanism. The Poisson process indicates that the rate at which clients enter the system is λ_1. Clients require services in both systems, which are separated by the batch service process. The largest server size is 2560 GB, while the minimum is 64 MB. Customers' data entry triggers a data search in the list, which starts the FES. The RDX mechanism acts as a form of service and is used to complete the first node search process.

Furthermore, SES starts with the setting of the IBM WebSphere data, and at the end of the process, which is improved by AI, the confirmation of all data is obtained. The server switches to a secondary activity, which can involve updating the system or deleting temporary files, once a service is complete and no further study has to be performed.

Let X be the group size random variable of the arrival. λ_1 is the Poisson arrival rate, g_k is the probability that '*k*' customers arrive in a batch, and X(z) is the PGF of X. In this study, the cumulative distribution function (CDF), probability density function (PDF), and remaining service time (RST) are studied using the Laplace–Stieltjes transform (LST) for varying batch service. Table 1 presents the notations for the cases listed above.

Table 1. Notations.

	CDF	PDF	LST	RST
Batch Service	$B(x)$	$b(x)$	$\widetilde{B}(\theta)$	$B^0(x)$
Vacation	$Q(x)$	$q(x)$	$\widetilde{Q}(\theta)$	$Q^0(x)$

The number of clients waiting for service at time *t* is denoted with $N_q(t)$. The number of clients using the service at the moment *t* is $N_s(t)$. It is defined as follows:

$$A(t) = \begin{cases} 0, \textit{ when server is busy with varying batch service} \\ 1, \textit{when the server is on vacation} \\ 2, \textit{when the server is on dormant period} \end{cases}.$$

Now, it is possible to define the state probabilities as follows:

$$P_{ij}(x,t)dt = P_r\left\{ \begin{array}{c} N_s(t) = i,\ N_q(t) = j \\ x \le B^0(t) \le x + dt,\ A(t) = 0 \end{array} \right\};\ a \le i \le b;\quad j \ge 0;$$

is the probability that at time *t*, the server is busy with *i* customers under service and *j* customers in the queue, and the remaining service time of a customer under service is between *x* and *x* + *dt*.

$$Q_n(x,\ t)dt = P_r\left\{ \begin{array}{c} N_q(t) = n,\ x\ \le\ Q^0(t)\ \le\ x + dt \\ A(t) = 1, 0 \le\ n\ \le\ a - 1 \end{array} \right\};$$

the probability that at time *t* the server is on vacation, the queue length is *n*, and the remaining vacation time of a customer at an arbitrary time is between x and

$$T_n(t) = P_r\{N_q(t) = n, A(t) = 2\}, 0 \le n \le a - 1,$$

is the probability that at time t the server is in the dormant period and the queue length is *n*.

In the following, a steady-state analysis of such a system is conducted. The supplemental variable technique, used in this analysis, offers the following relations:

$$-\frac{d}{dx}P_{i0}(x) = -\lambda_1 P_{i0}\ (x) + \sum_{m=0}^{a-1} T_m\ \lambda_1 g_{i-m} b(x) + \sum_{m=a}^{b} P_{mi}(0)b(x) + Q_i(0)b(x), a \le i \le\ \ b; \tag{1}$$

Equation (1) denotes the different probabilities for the server in a batch service with 'i' customers in service and '0' customers in the queue in the remaining service time $x - \Delta t$ at time t + Δt. In RHS, the first term indicates that there are no arrivals at that time; the next term indicates that during the dormant period when 'i-m' customers arrived in the system, then the service will be initiated with 'i' customers going for batch service and '0' customers waiting in the queue; the third term indicates that, when batch service is completed, if '*i*' customers are in the queue, then 'i' customers go for batch service and '0' customers are waiting in the queue; and the last term indicates that after vacation completion, if '*i*' customers are in the queue, then 'i' customers go for batch service and '0' customers will be waiting in the queue. Similarly, Equations (2)–(8) can be expressed with the help of a schematic representation of the queueing model from Figure 1.

$$-\frac{d}{dx}P_{ij}(x) = -\lambda_1 P_{ij}(x) + \sum_{k=1}^{j} P_{i\ j-k}(x)\ \ \lambda_1 g_k\ , \mathrm{a} \le \mathrm{i} \le \mathrm{b}-1,\ \ \ \mathrm{j} \ge 1; \tag{2}$$

$$\begin{aligned} -\tfrac{d}{dx}P_{bj}(x) = \text{“} -\lambda_1 P_{bj}(x) + \textstyle\sum_{k=1}^{j} P_{b\ j-k}(x)\ \ \lambda_1 g_k + \sum_{m=a}^{b} P_{mb+j}(0)b(x) + \sum_{m=0}^{a-1} T_m\ \lambda_1 g_{b+j-m} b(x) \\ +Q_{b+j}(0)b(x),\ \mathrm{j} \ge 1; \end{aligned} \tag{3}$$

$$-\frac{d}{dx}Q_0(x) = -\lambda_1 Q_0(x) + \sum_{m=a}^{b} P_{m0}(0)\ \mathrm{q(x)} \tag{4}$$

$$-\frac{d}{dx}Q_n(x) = -\lambda_1 Q_n(x) + \sum_{m=a}^{b} P_{mn}(0)q(x) + \sum_{k=1}^{n} Q_{n-k}(x)\ \lambda_1 g_k,\ \ 1 \le \mathrm{n} \le \mathrm{a}-1; \tag{5}$$

$$\begin{aligned} -\tfrac{d}{dx}Q_n(x) = -\lambda_1 Q_n(x) + \textstyle\sum_{m=a}^{b} P_{mn}(0)q(x) + \sum_{k=1}^{n} Q_{n-k}(x)\ \ \lambda_1 g_k, \\ n = a-1; \end{aligned} \tag{6}$$

$$0 = -\lambda_1 T_0 +\ Q_0(0)\ ; \tag{7}$$

$$0 = -\lambda_1 T_n + Q_n(0) + \sum_{k=1}^{n} T_{n-k}\ \lambda_1 g_k\ \ , 1 \le \mathrm{n} \le \mathrm{a}-1. \tag{8}$$

Applying the Laplace–Stieltjes transform, we obtain $\widetilde{\mathrm{P}}_{\mathrm{in}}(\theta)$ and $\widetilde{\mathrm{Q}}_{\mathrm{n}}(\theta)$:

$$\widetilde{\mathrm{P}}_{\mathrm{ij}}(\theta) = \int_0^{\infty} \mathrm{e}^{-\theta \mathrm{x}} \mathrm{P}_{\mathrm{ij}}(x)dx,\ \widetilde{\mathrm{Q}}_{\mathrm{n}}(\theta) = \int_0^{\infty} \mathrm{e}^{-\theta \mathrm{x}} \mathrm{Q}_{\mathrm{n}}(\mathrm{x})\mathrm{dx}$$

$$\begin{aligned} \theta\widetilde{\mathrm{P}}_{\mathrm{i0}}(\theta) - P_{i0}(0) = \lambda_1 \widetilde{P}_{io}(\theta) - Q_i(0)\widetilde{B}(\theta) - \textstyle\sum_{m=a}^{b} P_{mi}(0)\widetilde{B}(\theta) \\ -\widetilde{B}(\theta)\textstyle\sum_{m=0}^{a-1} T_m\ \lambda_1 g_{i-m}, a \le i \le\ \ b; \end{aligned} \tag{9}$$

$$\theta\widetilde{\mathrm{P}}_{\mathrm{ij}}(\theta) - P_{ij}(0) = \lambda_1 \widetilde{P}_{ij}(\theta) - \sum_{k=1}^{j} \widetilde{P}_{i\ j-k}(\theta)\ \ \lambda_1 g_k, \mathrm{a} \le i \le \mathrm{b}-1,\ \mathrm{j} \ge 1; \tag{10}$$

$$\begin{aligned} \theta\widetilde{\mathrm{P}}_{\mathrm{bj}}(\theta) - P_{bj}(0) = \lambda_1 \widetilde{P}_{bj}(\theta) - \textstyle\sum_{k=1}^{j} \widetilde{P}_{b\ j-k}(\theta)\ \ \lambda_1 g_k - \widetilde{B}(\theta) \\ \textstyle\sum_{m=a}^{b} P_{mb+j}(0) - \widetilde{B}(\theta)\sum_{m=0}^{a-1} T_m\ \lambda_1 g_{b+j-m} - \widetilde{B}(\theta)Q_{b+j}(0); \end{aligned} \tag{11}$$

$$\theta\widetilde{\mathrm{Q}}_0(\theta) - Q_o(0) = \lambda_1 \widetilde{Q}_0(\theta) - \widetilde{Q}(\theta)\sum_{m=a}^{b} P_{m0}(0) \tag{12}$$

$$\theta\widetilde{\mathrm{Q}}_{\mathrm{n}}(\theta) - Q_n(0) = \lambda_1 \widetilde{Q}_n(\theta) - \sum_{k=1}^{n} \widetilde{Q}_{n-k}(\theta)\ \lambda_1 g_k - \widetilde{Q}(\theta)\sum_{m=a}^{b} P_{mn}(0), 1 \le \mathrm{n} \le \mathrm{a}-1; \tag{13}$$

$$\theta\widetilde{\mathrm{Q}}_{\mathrm{n}}(\theta) - Q_n(0) = \lambda_1 \widetilde{Q}_n(\theta) - \sum_{k=1}^{n} \widetilde{Q}_{n-k}(\theta)\ \lambda_1 g_k - \widetilde{Q}(\theta)\sum_{m=a}^{b} P_{mn}(0), \mathrm{n} = \mathrm{a}-1. \tag{14}$$

2.1. Probability-Generating Function (PGF)

The following probability-generating functions are defined in order to obtain the probability-generating function (PGF) of the queue size at any given time epoch.

$$\widetilde{P}_i(z,\theta) = \sum_{j=0}^{\infty} \widetilde{P}_{i,j}(\theta) z^j, \ \ P_i(z,0) = \sum_{j=0}^{\infty} P_{ij}(0) z^j$$

$$\widetilde{Q}(z,\theta) = \sum_{n=0}^{a-1} \widetilde{Q}_n(\theta) z^n, \ \ Q(z,0) = \sum_{n=0}^{a-1} Q_n(0) z^n \tag{15}$$

$$T(z) \sum_{n=0}^{a-1} T_n z^n \quad a \le i \le b$$

After summing over n and multiplying the equations from (9)–(14) by the appropriate powers of z^n, we can use (15) to obtain the probabilities.

Consider now $P\ (z)$ as a PGF at an arbitrary time. This is defined by:

$$P\ (z) = \sum_{m=a}^{b-1} \widetilde{P}_m(z,0) + \widetilde{P}_b(z,\ 0) + \widetilde{Q}\ (z,\ 0) + T\ (z). \tag{16}$$

By multiplying Equations (9)–(14) with corresponding powers of z (under the form z^n) and making the sum over n, taking into account Equation (16), we obtain:

$$(\theta - \lambda_1 + \lambda_1 X(z))\widetilde{P}_i(z,\theta) = P_i(z,0) - \widetilde{B}(\theta)\left\{\sum_{m=0}^{a-1} T_m \lambda_1 g_{i-m} + Q_i(0) + \sum_{m=a}^{b} P_{mi}(0), \right\} \\ a \le i \le b-1\ ; \tag{17}$$

$$z^b(\theta - \lambda_1 + \lambda_1 X(z))\widetilde{P}_b(z,\theta) = (z^b - \widetilde{B}(\theta))P_b(z,0) - \widetilde{B}(\theta)\left(\sum_{m=c}^{b-1} P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0) z^j\right) \\ -\widetilde{B}(\theta)(Q(z,0) - \sum_{n=0}^{b-1} Q_n(0) z^n) - \widetilde{B}(\theta)\left[\lambda_1\left(T(z)X(z) - \sum_{m=0}^{c-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)\right)\right] \tag{18}$$

$$(\theta - \lambda_1 + \lambda_1 X(z))\widetilde{Q}(z,\theta) = Q(z,0) - \widetilde{Q}(\theta)\sum_{m=a}^{b} P_{mn}(0) z^n; 1 \le n \le a-1;$$

After performing some algebra and substituting $\theta = \lambda_1 - \lambda_1 X(z)$ in Equations (17)–(19), the PGF of the queue size specified in (16) is simplified to the following equation:

$$P(z) = \frac{\left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))Ci + \left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)f(z) + (z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\left(\widetilde{Q}(\lambda_1 - \lambda_1 X(z)) - 1\right)P_n\ Z^n + (z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))(-\lambda_1 + \lambda_1 X(z))T(Z)}{(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\ (-\lambda_1 + \lambda_1 X(z))}. \tag{19}$$

In this formula, the following notations were used:

$$c_i = p_i + q_i + t_i,\ p_i = \sum_{m=a}^{b} P_{mi}(0), \quad q_i\ = Q_i(0)\ , t_i = \lambda_1 T_m\ g_{i-m}\ ,$$

$$f(z) = \widetilde{B}\ c_i - P_j z^j + \widetilde{Q} P_n z^n - q_n z^n + \lambda_1 E_1,\ P_n = \sum_{m=a}^{b} P_{mn}(0),$$

$$T(z) = \sum_{n=0}^{a-1} T_n z^n,\ E_1 = \left(T(z)X(z) - \sum_{m=0}^{c-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)\right),$$

$$P_j\ = \sum_{j=0}^{b-1}\ P_{mj}(0). \tag{20}$$

If 'ρ' is defined as $(b - \lambda_1 E(B)E(X))$, the condition $\rho < 1$ is satisfied in the case of the existence of a steady state in this system.

Using these preliminaries, in the following, the performance measure of node 1 can be determined.

Node 1:

Expected Queue Length

To determine the average number of customers at an arbitrary time t, we can use the following relation:

$$E(Q) = \lim_{z \to 1} P'(z) \; ;$$

or

$$E(Q) = \frac{u''' \;\; v'' - u'' \;\; v'''}{12(\lambda_1 E(X)(b - \lambda E(B)E(X)))^2} \tag{21}$$

In Equation (22), the following notations and Equation (21) have been used:

$$v = (z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\;(-\lambda_1 + \lambda_1 X(z)),$$

and

$$u = \begin{array}{c} \left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))Ci + \\ \left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right) f(z) + \\ (z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\left(\widetilde{Q}(\lambda_1 - \lambda_1 X(z)) - 1\right) P_n\; Z^n + \\ (z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))(-\lambda_1 + \lambda_1 X(z))T(Z) \end{array}$$

Expected Waiting Time in the Queue—E(W)

The average waiting time considering the queue (E (W)) is determined with Little's formula as follows:

$$E\;(W) = \frac{E(Q)}{\lambda_1 E(X)} \tag{22}$$

Expected Length of Idle Period—E(I)

The length of the idle period can be determined with the following relation:

$$E(I) = \frac{1}{\lambda_1} \sum\nolimits_{i=0}^{a-1} \alpha_{i,} \tag{23}$$

Here, $\frac{1}{\lambda_1}$ represents the expected staying time in the state 'i' in an idle period.

Expected Length of Busy Period—E(B_1)

For the busy time, let B_1 be the random variable. Define another random variable J as follows:

J = 0, if the server finds less than 'a' customers in the queue after a service.

J = 1, if the server finds at least 'a' customers in the queue after a service, where $E(B)$ is the expected service time.

This results in:

$$E\;(B_1\;) = \frac{E(B)}{P(J=0)} = \frac{E(B)}{\sum_{i-0}^{a-1} P_i}. \tag{24}$$

Now, an analysis of node 2 will be performed.

Let $A(z)$ be the PGF of the queue size at an arbitrary time for node 2 and X the group size random variable of the bulk arrival at node 2. Then:

$$\mathrm{A(Z)} = \frac{(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1)(z^b\; c_i - P_j \mathrm{z}^{\mathrm{j}} + \widetilde{Q} P_n \mathrm{z}^{\mathrm{n}} - q_n \mathrm{z}^{\mathrm{n}} + \lambda_1 E_1)}{(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\;((-\lambda_1 + \lambda_1 X(z)) - \left(\widetilde{Q}(\lambda_1 - \lambda_1 X(z)) - 1\right) P_n\; Z^n - (-\lambda_1 + \lambda_1 X(z)))T(Z))}. \tag{25}$$

Node 2:

Expected Queue Length—E(U):

It is possible to determine the average number of customers:

$$E(U) = \lim_{z \to 1} A'(z)$$

$$E(U) = \frac{\lambda_1 E(X)E(B)v_2}{12(\ b - \lambda_1 E(B)E(X))(\lambda_1 E(X) - \lambda_1 E(Q)E(X)P_n - E(X)T_n)}; \tag{26}$$

with $v_2 = P_j - c_i - P_n + q_n - \lambda_1 E_1$.

Expected Waiting Time in the Queue—E(V)

It is now possible to use Little's formula to determine the mean waiting time of the customers in the queue, $E(V)$:

$$\mathrm{E}(V) = \frac{E(U)}{\lambda_1 E(X)}. \tag{27}$$

2.2. Particular Case

In this section, some of the PGFs of the existing models are deduced as a particular case of the proposed model.

Case 1: When there is no vacation in node 1 (i.e., $\widetilde{Q}(\lambda_1 - \lambda_1 X(z)) = 1$).

$$P(z) = \frac{\left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))Ci + \left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)f(z) + +(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))(-\lambda_1 + \lambda_1 X(z))T(Z)}{(z^b - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\ (-\lambda_1 + \lambda_1 X(z))} \tag{28}$$

In this formula, the following notations were used:

$$c_i = p_i + t_i,\ \mathrm{p_i} = \sum\nolimits_{m=a}^{b} P_{mi}(0)\ , t_i = \lambda_1 T_m\ g_{i-m}\ ,$$

$$f(z) = \widetilde{B}\ c_i - P_j \mathrm{z^j} + \lambda_1 E_1,\ P_n = \sum\nolimits_{m=a}^{b} P_{mn}(0),$$

$$\mathrm{T(z)} = \sum\nolimits_{n=0}^{a-1} T_n z^n,\ E_1 = \left(T(z)X(z) - \sum\nolimits_{m=0}^{c-1}\left(\mathrm{T_m z^m}\sum\nolimits_{\mathrm{j=1}}^{\mathrm{b-m-1}} \mathrm{g_j z^j}\right)\right),$$

$$P_j\ = \sum\nolimits_{j=0}^{b-1}\ P_{mj}(0).$$

Case 2: When the batch service mode changes to single-service mode (i.e., *a* = *b* = 1). The PGF given in Equation (20) is reduced to

$$P(z) = \frac{\left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)(z - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))Ci + \left(\widetilde{B}(\lambda_1 - \lambda_1 X(z)) - 1\right)f(z) + (z - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\left(\widetilde{Q}(\lambda_1 - \lambda_1 X(z)) - 1\right)P_n\ Z^n + (z - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))(-\lambda_1 + \lambda_1 X(z))T(Z)}{(z - \widetilde{B}(\lambda_1 - \lambda_1 X(z)))\ (-\lambda_1 + \lambda_1 X(z))} \tag{29}$$

In this formula, the following notations were used:

$$c_i = p_i + q_i + t_i,\ \mathrm{p_i} = P_{1i}\ (0),\quad q_i\ = Q_i(0)\ , t_i = \lambda_1 T_m\ g_{i-m}\ ,$$

$$f(z) = \widetilde{B}\ c_i - P_j \mathrm{z^j} + \widetilde{Q} P_n \mathrm{z^n} - q_n \mathrm{z^n} + \lambda_1 E_1,\ P_n\ = P_{1n}(0),$$

$$\mathrm{T(z)} = \sum\nolimits_{n=0}^{\infty} T_n z^n,\ E_1 = \left(T(z)X(z) - \sum\nolimits_{m=0}^{\infty}\left(\mathrm{T_m z^m}\sum\nolimits_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{g_j z^j}\right)\right), P_j\ = P_{mo}(0).$$

The results are identical to those found in [28].

3. Cost Model

In many real-world situations, optimization strategies are essential for reducing a queuing system's total average cost. Cost analysis usually takes into consideration a number of limitations, including carrying expenses, in-service costs, switch-on costs, and possible compensation costs. Achieving the lowest possible total average cost is the main goal of system management. The cost analysis of the suggested queuing method is shown in this section, predicated on the following hypotheses:

γ_h : carrying cost per customer

γ_0 : In service cost per unit time

γ_s : switch on cost per cycle

γ_r : remuneration cost per cycle due to vacation

The length of the cycle is as follows:

$$E(T_c) = E(\text{Average sleep time of the server}) + E(\text{Average processing period of the server})$$

$$E(T_c) = \frac{1}{\tilde{\xi}} \sum_{i=0}^{a-1} \alpha_{i,} \;\; + \frac{E(B_1\ )}{\sum_{i-0}^{a-1} P_i},$$

and the total average cost (TAC) can be determined with the following relation:

$$\text{Total average cost} = [\gamma_s - \;\; \gamma_r E(I)] \frac{1}{E(T_c)} + \gamma_h E(Q) + \gamma_0 \rho \tag{30}$$

where $\rho = \frac{(\lambda_1 E(B) E(X))}{N}$

The best course of action for a threshold c* to minimize the aggregate mean cost is determined using the direct search approach.

Step 1: Determine the value of maximum batch size '*b*'.

Step 2: Choose the value '*c*' satisfying the condition: TAC(c*) $\leq$ TAC(c), $1 \leq c \leq b$.

Step 3: The value *c** is optimal, as it provides minimum TAC.

The ideal value of *c** that minimizes the TAC function is obtained using the preceding technique. The following section will provide a numerical example to support this solution.

4. Results

A presentation of the utility of the model and some numerical examples for illustrative purposes are described herein. The following notations are used to numerically justify the theoretical conclusions derived for the suggested model (Table 2):

Table 2. Parameters and notations.

Parameter	Distribution	Notations	PDF	CDF
Arrival rate	Poisson distribution	λ_1	$P(X=K) = \frac{e^{-\lambda_1} \quad \lambda_1{}^K}{K!}$ K = 0, 1, 2, 3.	$P(X \leq K) = \sum_{i=0}^{K} \frac{e^{-\lambda_1} \quad \lambda_1{}^i}{i!}$
Node 1 service time	2-Erlang distribution	μ_1	f_r(t) = $\mu_1{}^2 te^{-\mu_1 t}$ $t \geq 0$	F_r(t) = 1 − (1+$\mu_1 t)e^{-\mu_1 t}$ $t \geq 0$
Node 2 service time	2-Erlang distribution	Y	f_r(t) = $Y^2 te^{-Yt}$ $t \geq 0$	F_r(t) = 1 − (1+$Yt)e^{-Yt}$ $t \geq 0$
Vacation time	Exponential	ε_{ς}	f_r(t) = $\varepsilon_{\varsigma} e^{-\varepsilon_{\varsigma} t}$ $t \geq 0$	F_r(t) = 1− $e^{-\varepsilon_{\varsigma} t}$ $t \geq 0$

The effects of different performance metrics for a set of threshold values are shown. It is evident from Table 3 and Figure 2 that if λ_1 increases, *E* (*Q*), *E* (B_1), and *E* (*W*) rise and E (I) falls. Node 1 represents the data provided. It is evident from Table 4 and Figure 3 that if μ_1 increases, *E* (*Q*), *E* (B_1), and *E* (*W*) fall and E (I) rises. Node 1 represents the data provided. The effects of different performance metrics for a set of threshold values are shown. It is evident from Table 5 and Figure 4 that E (U) and E (V) increase in tandem with increases in λ_1 and Y. Table 5 uses node 2 to represent the data. The numerical results illustrate the impact of system parameters on performance, offering valuable insights for optimization. Table 6 and Figure 5 present the total average cost of the system. To minimize this cost, the lower bound server capacity should be set at $a = 5$. Additionally, the parameter 'a' should be optimized based on fluctuations in arrival, service, and vacation rates. For optimal performance, maintaining $a = 5$ and $b = 10$ is recommended. As shown in Figure 5, an increase in the threshold value 'a' and E (Q) leads to a decrease in the total average cost.

Table 3. Arrival rate versus performance measures. $E(Q)$—Expected queue length, $E\ (W)$—Expected waiting time in the queue, $E\ (I)$—Expected length of idle period, and $E\ (B)$—Expected length of busy period.

For $a = 2$, $b = 4$, $\varepsilon = 10$, $\mu_1 = 4$,				
λ_1	$E(Q)$	$E(B_1)$	$E(I)$	$E(W)$
2.2	6.835	3.345	0.187	1.654
2.4	8.673	4.653	0.156	1.678
2.7	9.462	4.712	0.145	1.764
2.8	11.764	5.875	0.085	1.868
3.0	13.341	5.962	0.076	1.975

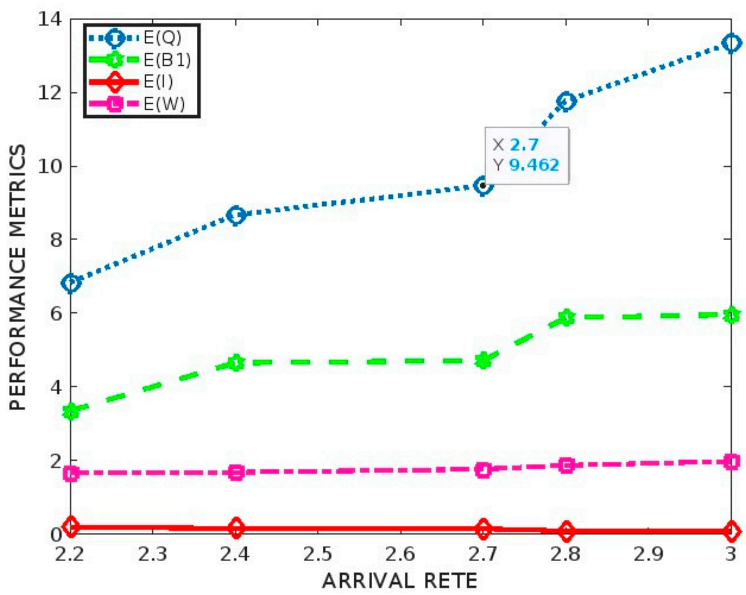


Figure 2. Arrival rate versus performance measures—node 1. *For* $a = 2$, $b = 4$, $\varepsilon = 10$, and $\mu_1 = 4$.

Table 4. Service rate versus performance measures.

For $a = 4$, $b = 7$, $\lambda_1 = 8$				
μ_1	$E(Q)$	$E(B_1)$	$E(I)$	$E(W)$
6.1	12.9576	11.3865	0.5248	8.5529
6.3	11.6945	10.0895	1.0542	7.6812
6.5	10.4678	9.2096	2.3567	6.3587
6.7	9.7754	8.6896	3.4324	4.9782
6.9	8.3986	7.9437	4.5375	3.4271

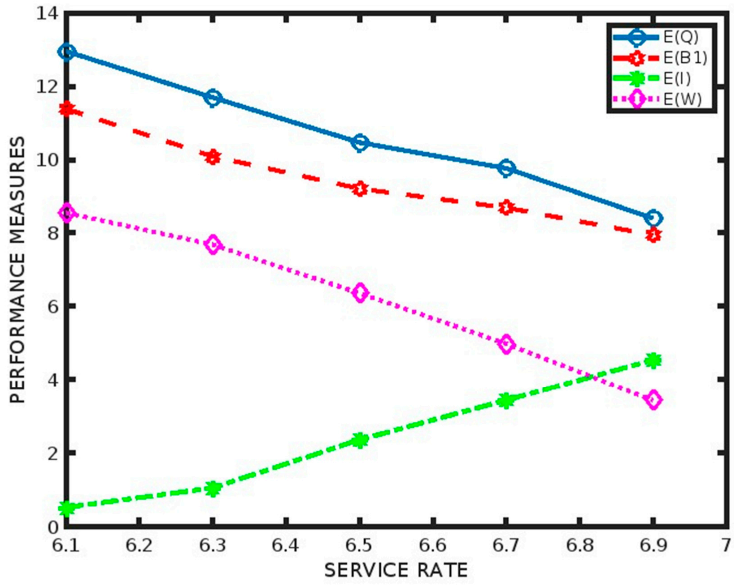


Figure 3. Arrival rate versus performance measures. For $a = 4$, $b = 7$, and $\lambda_1 = 8$.

Table 5. Arrival rate versus performance metrics of node 2. $E(U)$ = Expected length of queue for node 2. $E(V)$ = Expected waiting time in queue for node 2. For a = 3, b = 5, ε = 9, and Y = 4.

λ_1	$E(U)$	$E(V)$
4.2	6.5643	3.1237
4.4	7.1236	3.5973
4.8	8.5743	4.0124
5.2	10.1786	5.3458
5.8	11.6542	6.6735

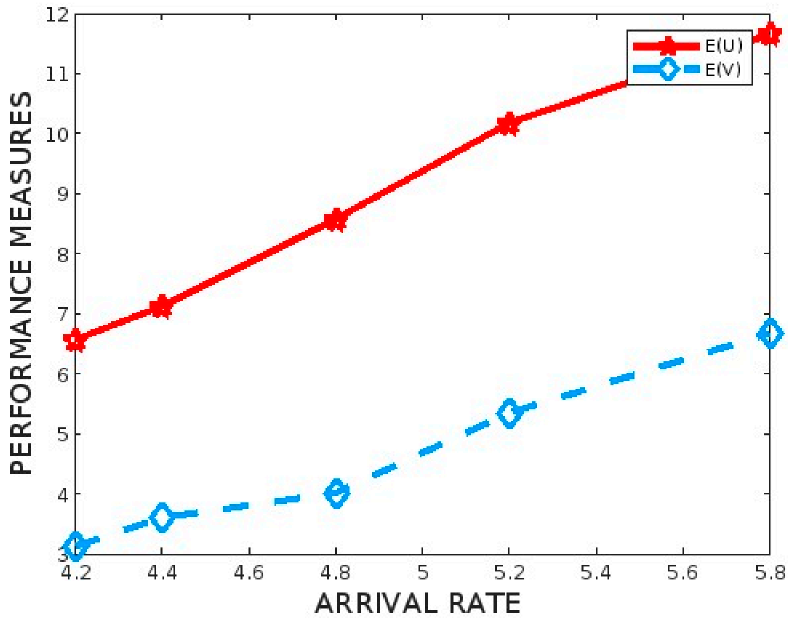
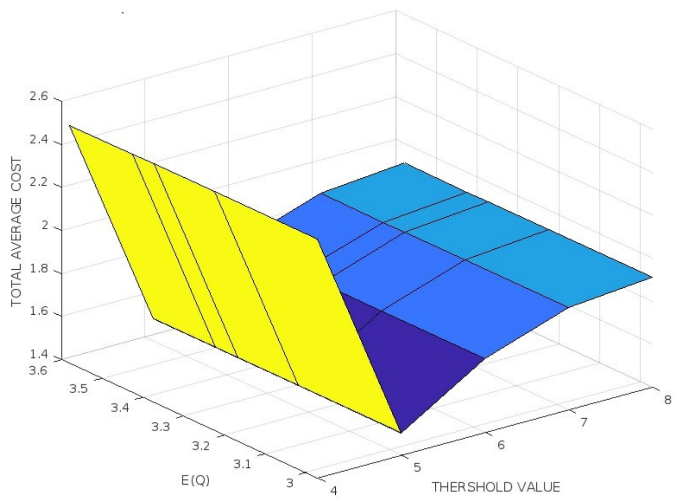


Figure 4. Arrival rate versus performance measures—node 2.

Table 6. Performance measures and total average cost (TAC) versus threshold value '*a*'. For $\lambda_1 = 6, b = 10, \mu_1 = 5, \varepsilon = 10$.

a	$E\ (Q)$	$E\ (B)$	$E\ (I)$	TAC
1	2.6251	6.1365	4.7265	3.3762
2	2.7624	6.3367	5.9327	3.1346
3	2.8293	6.4156	6.3594	2.8349
4	2.9724	7.9591	6.9725	2.5081
5	3.2253	8.9369	7.6921	1.5034
6	3.3726	9.3291	8.4392	1.7435
7	3.4267	9.8328	9.3982	1.8742
8	3.5789	9.9429	10.8328	1.9095



For $\boldsymbol{\lambda_1 = 6, b = 10}, \mu_1 = 5, \varepsilon = 10$

Figure 5. Threshold value '*a*', E (Q), and total average cost.

5. Conclusions

This study examines the $M^X/G\ (a, b)/1$ queueing model, which requires two stages of tandem queue service and takes vacations into account. The distinctive feature of the model proposed in this study is derived from the addition of vacations to the first node of the tandem queuing system. By using techniques involving additional variables, a queue-size generating function was developed. A numerical illustration of the method provides the key performance metrics of this model and tests its validity. In the future, this model could be extended to include many vacations and vacation interruptions. A cost analysis was also performed in this study that identified threshold values for different nodes. This analysis allows for the effective minimization of the average total cost of the system, which is extremely important in practical applications. All these considerations highlight the practical utility of the proposed model for the design and optimization of real-world queuing systems, especially those involving batch processing and vacation dynamics. One direction for further development is to improve this model by integrating more vacation policies, vacation interruptions, or priority-based service disciplines. In addition, examining multi-node systems with diverse arrival and service distributions could lead to further generalizations and practical applications. Such extensions would broaden the relevance of the model to a wider spectrum of operational and service environments.

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