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Generalized Iterated Function Systems on b-Metric Spaces

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Abstract: An iterated function system consists of a complete metric space (X, d) and a finite family of contractions $f_1, \ldots, f_n : X \to X$. A generalized iterated function system comprises a finite family of contractions defined on the Cartesian product X^m with values in X. In this paper, we want to investigate generalized iterated function systems in the more general setting of b-metric spaces. We prove that such a system admits a unique attractor and, under some further restrictions on the b-metric, it depends continuously on parameters. We also provide two examples of generalized iterated function systems defined on a particular b-metric space and find the corresponding attractors.

Keywords: fractals; iterated function systems; fixed points; attractor; b-metric spaces

MSC: 28A80; 37C70; 54H25

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1. Introduction

A central role in the study and generation of fractal sets is played by the concept of iterated function systems (IFSs), which was introduced in its present form in 1981 by Hutchinson [1] and popularized by Barnsley [2]. An IFS consists of a complete metric space (X, d) and a finite family of Banach contractions $f_1, \ldots, f_n : X \to X$. Such a system induces a set function, known as the Hutchinson operator, $F_S : P_{cp}(X) \to P_{cp}(X)$, which is defined by

$$F_S(B) = \bigcup_{k=1}^{n} f_k(B),$$

for all $B \in P_{cp}(X)$, where $P_{cp}(X)$ stands for the family of all nonempty and compact subsets of X. On the complete metric space formed by $P_{cp}(X)$ and the Hausdorff–Pompeiu metric h, the set function F_S is a Banach contraction; therefore, by the contraction mapping principle, it admits a unique fixed point, denoted by A_S, which is called the attractor of the IFS. Such attractors are also known as Hutchinson–Barnsley fractals. For very recent research in this direction, see [3]. Iterated function systems have applications in various domains such as engineering sciences, medicine, forestry, economy, human anatomy, physics, and especially in fractal image compression.

In an effort to extend the theory of fractal sets, in 2008, Miculescu and Mihail [4,5] introduced the concept of generalized iterated function system (GIFS) of order m, which comprises a finite family of Banach contractions defined on the finite Cartesian product X^m with values in X. They proved that such a system has a unique attractor and studied several of its properties. In 2012, Strobin and Swaczyna [6] extended these results to the more general case of a GIFS consisting of φ-contractions rather than Banach contractions. The GIFS turns out to be an effective generalization of the classical IFS, since Strobin [7] proved that for any $m \geq 2$, there exists a Cantor subset of the plane which is an attractor of some GIFS of order m, but it is not the attractor of any GIFS of order $m - 1$.

We present a further generalization of the Hutchinson–Barnsley theory of iterated function systems by considering GIFSs consisting of φ-contractions on b-metric spaces. Our purpose is to see if the results concerning the attractors of GIFSs on metric spaces from [6] can be extended to this more general setting. Known as quasimetric spaces (see [8]), b-metric spaces represent a generalization of the metric spaces obtained by relaxing the triangle inequality. Every metric space is a b-metric space, but the converse is not true. The sequence space $l^p(\mathbb{R})$ under the b-metric $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to [0, \infty)$, given by

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

for all $x = (x_n)_n, y = (y_n)_n \in l^p(\mathbb{R})$ and the Lebesgue space $L^p([0,1])$ under the b-metric $d : L^p([0,1]) \times L^p([0,1]) \to [0, \infty)$, given by

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}},$$

for all $x = x(t), y = y(t) \in L^p([0,1])$, when $p \in (0,1)$, are classical examples of b-metric spaces that are not metric.

In recent years, an intensive study of b-metric spaces has been carried out, which was mainly concentrated on transposing various topological properties and fixed point results to this framework (see, for example, Refs. [9–12]). A great survey on the origins and early developments of b-metric spaces can be found in [8]. One can endow a b-metric space with a topology in the usual way, but it is worth mentioning that in this setting, the distance function may not be continuous and open balls are not necessarily open sets.

In this paper, we prove that a GIFS admits a unique attractor (Theorem 5) and, under certain conditions, it depends continuously on parameters (Theorem 7) when it is defined on a b-metric space. Our results fall within the line of research that aims to expand the class of attractors of iterated function systems by adopting a wider framework for the spaces on which the contractions are defined. We provide two examples of GIFSs defined on a concrete b-metric space and find the corresponding attractors.

2. Preliminaries and Definitions

In this section, we present the notations and some of the standard facts on b-metric spaces. We also introduce the notion of a GIFS in the context of b-metric space.

Definition 1. *Let X be a nonempty set and $s \geq 1$. We say that a function $d : X \times X \to [0, \infty)$ is a b-metric if it satisfies the following properties:*

(i) $d(x,y) = 0$ *if and only if* $x = y$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$,

for all $x, y, z \in X$.

The triplet (X, d, s) is called a b-metric space.

The third condition (iii) is called the s-relaxed triangle inequality.

Remark 1. *Every metric space is a b-metric space (with $s = 1$), but the converse is not true. There exist b-metric spaces that are not metric.*

Example 1 ([9]). *Let* $X = \left\{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\}$ *and* $d : X \times X \to [0, \infty)$ *be defined by*

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \in \{0,1\} \\ |x-y| & if\ x \neq y \in \{0\} \cup \left\{\frac{1}{2n} : n = 1, 2, \dots\right\} \\ 4 & otherwise. \end{cases}$$

Then, d is a b-metric on X with $s = \frac{8}{3}$, but it is not a metric on X.

Definition 2. *Let (X, d, s) be a b-metric space. A sequence of elements $(x_n)_n \subseteq X$ is said to be*

(i) *convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$;*

(ii) *Cauchy if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$, i.e., for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m \in \mathbb{N}$ with $n, m \geq N_\varepsilon$.*

We say that (X, d, s) is a complete b-metric space if every Cauchy sequence from (X, d, s) is convergent.

Definition 3. *Let (X, d, s) and (Y, ρ, r) be two b-metric spaces. A function $f : X \to Y$ is continuous if for every $(x_n)_n \subseteq X$ and $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$, we have $\lim\limits_{n\to\infty} f(x_n) = f(x)$.*

Even though in a b-metric space the distance function may fail to be continuous and open sets may not be open (see [9]), many of the topological properties of sequences and sets typical of metric spaces remain valid in this more general setting.

Proposition 1. *Let (X, d, s) be a b-metric space and $A \subseteq X$. If we define $\overline{A}$ to be the intersection of all closed subsets of X, then $x \in \overline{A}$ if and only if there exists a sequence $(x_n)_n \subseteq A$ such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$.*

Theorem 1. *Let (X, d, s) be a b-metric space and $A \subseteq X$ be a nonempty subset.*

(i) *A is compact if and only if A is sequentially compact.*

(ii) *If A is compact, then A is totally bounded.*

Since in a b-metric space, the distance function need not be continuous, for the second part of our paper, we need the following concept.

Definition 4. *Let (X, d, s) be a b-metric space. The b-metric d is called lower semicontinuous if for any $(x_n)_n, (y_n)_n \subseteq X$ and $x, y \in X$ such that $d(x_n, x) \underset{n}{\to} 0, d(y_n, y) \underset{n}{\to} 0$, it follows that*

$$d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n).$$

Let (X, d, s) and (Y, ρ, r) be two b-metric spaces and $f : X \to Y$. The Lipschitz constant of f is, by definition,

$$\text{Lip}(f) = \inf\{c > 0 : \rho(f(x), f(y)) \leq c d(x,y); x, y \in X\}.$$

If $\text{Lip}(f) \leq 1$, then f is said to be nonexpansive.

For a function $f : X \to X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we denote the composition of f by itself n times. By $f^{[0]}$, we mean the identity function $Id_X : X \to X$.

We say that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Browder comparison function if φ is right continuous, nondecreasing and $\varphi(t) < t$ for any $t > 0$.

Remark 2 ([13]). *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Browder comparison function, then φ is upper semicontinuous, i.e., for any $x_0 > 0$ and $(x_n)_n \subseteq \mathbb{R}_+$ such that $\lim\limits_{n\to\infty} x_n = x_0$, we have $\limsup\limits_{n\to\infty} \varphi(x_n) \leq \varphi(x_0)$.*

Definition 5. *Let (X, d, s) and (Y, ρ, r) be two b-metric spaces. A function $f : X \to Y$ is called a φ-contraction if there exists a Browder comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\rho(f(x), f(y)) \leq \varphi(d(x, y))$$

for all $x, y \in X$.

The following result is a known fixed point theorem for b-metric spaces.

Theorem 2 ([10]). *Let (X, d, s) be a complete b-metric space. If $f : X \to X$ is a φ-contraction, then f is Picard, i.e., f has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} f^{[n]}(x) = x^*$ for all $x \in X$.*

As in the metric case, one can easily prove the following lemma.

Lemma 1. *Let (X, d, s) be a complete b-metric space and $f : X \to X$. Suppose that there exists $p \in \mathbb{N}$ such that $f^{[p]}$ is Picard. Then, f is Picard.*

Proof. Since $f^{[p]}$ is Picard, there exists a unique $x^* \in X$ such that $f^{[p]}(x^*) = x^*$ and for any $x \in X$, we have

$$\lim_{n\to\infty} \left(f^{[p]}\right)^{[n]}(x) = x^*.$$

Then,

$$f(x^*) = f^{[p+1]}(x^*) = f^{[p]}(f(x^*)),$$

so $x^* = f(x^*)$.

If there exists $y^* \in X$ such that $y^* = f(y^*)$, then

$$y^* = f(y^*) = f^{[2]}(y^*) = \ldots = f^{[p]}(y^*),$$

hence, x^* is the unique fixed point of f.

For any $i \in \{0, 1, \ldots, p-1\}$ and $x \in X$, we have

$$\lim_{k\to\infty} f^{[pk+i]}(x) = \lim_{k\to\infty} \left(f^{[p]}\right)^{[k]}(f^{[i]}(x)) = x^*.$$

□

Throughout this paper, we will write $P_{cp}(X)$ for the family of all nonempty and compact subsets of the b-metric space (X, d, s). The Hausdorff–Pompeiu b-metric on $P_{cp}(X)$ is defined by

$$h(D, G) = \max\left\{\sup_{x\in D} \inf_{y\in G} d(x, y), \sup_{y\in G} \inf_{x\in D} d(x, y)\right\},$$

for all $D, G \in P_{cp}(X)$.

It is known that if (X, d, s) is complete, then $(P_{cp}(X), h, s)$ is a complete b-metric space.

Proposition 2 ([14]). *Let (X, d, s) be a b-metric space. For all $(A_i)_{i\in I}, (B_i)_{i\in I} \subseteq P_{cp}(X)$ such that $\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i \in P_{cp}(X)$, we have*

$$h\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \leq \sup_{i\in I} h(A_i, B_i).$$

Let (X,d,s) be a b-metric space and $m \in \mathbb{N}$. It can be readily verified that the Cartesian product space $X^m = \underbrace{X \times X \times \ldots \times X}_{m}$ is a b-metric space with the same constant s under the maximum distance function:

$$d_{max}((x_1,\ldots,x_m),(y_1,\ldots,y_m)) = \max\{d(x_1,y_1),\ldots,d(x_m,y_m)\},$$

for all $(x_1,\ldots,x_m),(y_1,\ldots,y_m) \in X^m$.

We mention the following known result concerning the existence and uniqueness of fixed points for mappings defined on Cartesian products of metric spaces.

Theorem 3 ([15])**.** *Let (X,d) be a complete metric space and $f : X^m \to X$ such that*

$$d(f(x_1,..,x_m), f(x_2,\ldots,x_{m+1})) \le q_1 d(x_1,x_2) + \ldots + q_m d(x_m,x_{m+1}),$$

for all $x_1,\ldots,x_{m+1} \in X$, where $q_1,\ldots,q_m \ge 0$ such that $q_1 + \ldots + q_m \le 1$.
Then, there exists a unique $x^* \in X$ such that $f(x^*,\ldots,x^*) = x^*$. Moreover, for every $x_1,\ldots,x_m \in X$, the following sequence of iterates

$$x_{k+m} = f(x_k,\ldots,x_{k+m-1}), k \in \mathbb{N}$$

converges to x^*.

For the connection between multidimensional fixed point theorems and the classical fixed point theorems, one can consult [16] and [17].

Definition 6. *Let (X,d,s) be a b-metric space, $m,n \in \mathbb{N}$ and consider a finite family of φ-contractions $f_1,\ldots,f_n : X^m \to X$. The pair $S = ((X,d,s),\{f_1,\ldots,f_n\})$ is called a GIFS of order m on X.*

Note that if, for each $i \in \{1,\ldots,n\}$, f_i is a φ-contraction with a function φ_i, then it is also a φ-contraction with $\varphi := \max\{\varphi_1,\ldots,\varphi_n\}$.

One can associate with a GIFS S, a set function $F_S : P_{cp}(X)^m \to P_{cp}(X)$, also known as the Hutchinson operator, given by

$$F_S(D_1,\ldots,D_m) = \bigcup_{k=1}^{n} f_k(D_1 \times \ldots \times D_m) = \bigcup_{k=1}^{n} \{f(x_1,\ldots,x_m) : x_i \in D_i, i \in \{1,\ldots,m\}\},$$

for all $D_1,\ldots,D_m \in P_{cp}(X)$.

3. Results

In this section, our main results are stated and proved.

3.1. The Existence of the Attractor

This subsection is devoted to the study of the existence and uniqueness of attractors for GIFSs on b-metric spaces. The proofs are similar in spirit to those given in [6].

Lemma 2. *Let (X,d,s) be a b-metric space and $x^i, y^i \in X^m$ for $i \in \{1,\ldots,m\}$. If $f : X^m \to X$ is a φ-contraction, then*

$$d_{max}\Big(\big(f(x^1),\ldots,f(x^m)\big),\big(f(y^1),\ldots,f(y^m)\big)\Big) \le \varphi\Big(\max\Big\{d_{max}(x^1,y^1),\ldots,d_{max}(x^m,y^m)\Big\}\Big).$$

Proof. For any $i \in \{1,\ldots,m\}$, since f is a φ-contraction and φ is nondecreasing, we have

$$d(f(x^i),f(y^i)) \le \varphi(d_{max}(x^i,y^i)) \le \varphi\Big(\max\Big\{d_{max}(x^1,y^1),\ldots,d_{max}(x^m,y^m)\Big\}\Big).$$

Hence,

$$\max\Big\{d(f(x^1),f(y^1)),\dots,d(f(x^m),f(y^m))\Big\}\le\varphi\Big(\max\Big\{d_{max}(x^1,y^1),\dots,d_{max}(x^m,y^m)\Big\}\Big).$$

□

Let (X,d,s) be a b-metric space. Consider a mapping $f:X^m\to X$ and let $k:X^m\to X^m$ be given by

$$k(x_1,\dots,x_m)=(x_2,\dots,x_m,f(x_1,\dots,x_m))$$

for all $x_1,\dots,x_m\in X$.

It can be easily verified that k is nonexpansive whenever f is nonexpansive.

Theorem 4. *Let (X,d,s) be a complete b-metric space. If $f:X^m\to X$ is a φ-contraction, then f has a unique fixed point, i.e., there exists a unique $\alpha\in X$ such that $\alpha=f(\alpha,\dots,\alpha)$. Moreover, if for $x_1,\dots,x_m\in X$, we let $x_{k+m}=f(x_k,\dots,x_{k+m-1})$, $k\in\mathbb{N}$, then the sequence $(x_k)_k$ converges to α.*

Proof. Let $g:X\to X$ be the map defined by $g(x)=f(x,\dots,x)$ for all $x\in X$. Then, since g is a φ-contraction, by Theorem 2, we infer that there exists a unique $\alpha\in X$ such that

$$\alpha=g(\alpha)=f(\alpha,\dots,\alpha).$$

We see that f is nonexpansive, since $d(f(\overline{x}),f(\overline{y}))\le\varphi(d_{max}(\overline{x},\overline{y}))<d_{max}(\overline{x},\overline{y})$, for all $\overline{x},\overline{y}\in X^m$ with $\overline{x}\neq\overline{y}$. Hence, k is also nonexpansive.

Consider $\overline{x}=(x_1,\dots,x_m),\overline{y}=(y_1,\dots,y_m)\in X^m$. Let $x^1=\overline{x},y^1=\overline{y}$ and for $i\in\{1,\dots,m-1\}$, define

$$x^{i+1}=k(x^i),y^{i+1}=k(y^i).$$

Note that

$$x^2=k(x^1)=k(x_1,\dots,x_m)=(x_2,\dots,x_m,f(x^1)),$$
$$x^3=k(x^2)=k(x_2,\dots,x_m,f(x^1))=(x_3,\dots,x_m,f(x^1),f(x^2)),$$

so, by induction, we obtain

$$x^{i+1}=(x_{i+1},\dots,x_m,f(x^1),\dots,f(x^i)),$$

and similarly,

$$y^{i+1}=(y_{i+1},\dots,y_m,f(y^1),\dots,f(y^i))$$

for all $i\in\{1,\dots,m-1\}$. We see that

$$k^{[m]}(\overline{x})=k^{[m-1]}(k(x^1))=k^{[m-1]}(x^2)=k^{[m-2]}(k(x^2))=k^{[m-2]}(x^3)=\dots=k(x^m),$$

so $k^{[m]}(\overline{x})=k(x^m)=\big(f(x^1),\dots,f(x^m)\big)$ and likewise $k^{[m]}(\overline{y})=k(y^m)=\big(f(y^1),\dots,f(y^m)\big)$. From the fact that k is nonexpansive, it follows that

$$d_{max}(x^i,y^i)=d_{max}(k(x^{i-1}),k(y^{i-1}))\le\dots\le d_{max}(\overline{x},\overline{y}),$$

therefore, in view of Lemma 2, we deduce that

$$d_{max}(k^{[m]}(\overline{x}),k^{[m]}(\overline{y}))=d_{max}\Big(\Big(f(x^1),\dots,f(x^m)\Big),\Big(f(y^1),\dots,f(y^m)\Big)\Big)\le$$
$$\le\varphi(\max\Big\{d_{max}(x^1,y^1),\dots,d_{max}(x^m,y^m)\Big\})\le\varphi(d_{max}(\overline{x},\overline{y})),$$

so $k^{[m]}$ is a φ-contraction and thus a Picard operator. By Lemma 1, this implies that k is Picard. Therefore, there exists uniquely $(\beta_1, \dots, \beta_m) \in X^m$ such that

$$(\beta_1, \dots, \beta_m) = k(\beta_1, \dots, \beta_m) = (\beta_2, \dots, \beta_m, f(\beta_1, \dots, \beta_m)),$$

so $\beta_1 = \dots = \beta_m = \alpha = f(\alpha, \dots, \alpha)$.

Using the definition of $(x_k)_k$, we deduce that

$$k(x_1, \dots, x_m) = (x_2, \dots, x_m, x_{m+1})$$

$$k^{[2]}(x_1, \dots, x_m) = (x_3, \dots, x_{m+1}, x_{m+2})$$

and

$$k^{[n-1]}(x_1, \dots, x_m) = (x_n, \dots, x_{n+m-2}, x_{n+m-1})$$

for $n \in \mathbb{N}$. Finally, since k is Picard, it then follows that

$$(\alpha, \dots, \alpha) = \lim_{n\to\infty} k^{[n-1]}(x_1, \dots, x_m) = \lim_{n\to\infty} (x_n, \dots, x_{n+m-1}),$$

and we conclude that $\lim_{n\to\infty} x_n = \alpha$. □

Lemma 3. *Let (X, d, s) and (Y, ρ, r) be two b-metric spaces. If $f : X \to Y$ is a φ-contraction, then the set function $F_f : P_{cp}(X) \to P_{cp}(Y)$ given by $F_f(D) = f(D)$ for any $D \in P_{cp}(X)$ is also a φ-contraction (with the same φ), i.e.,*

$$h(f(D), f(G)) \le \varphi(h(D, G)),$$

for all $D, G \in P_{cp}(X)$.

Proof. Let us first show that $\varphi(\inf A) = \inf(\varphi(A))$ for any bounded subset $A \subseteq \mathbb{R}$. This follows from the following two observations.

By the definition of infimum, there exists a sequence $(u_n)_n \subseteq A$ such that $\lim_{n\to\infty} \varphi(u_n) = \inf(\varphi(A))$. Since $\inf A \le u_n$ and φ is nondecreasing, we have that $\varphi(\inf A) \le \varphi(u_n)$ for all $n \in \mathbb{N}$, therefore $\varphi(\inf A) \le \lim_{n\to\infty} \varphi(u_n)$, and

$$\varphi(\inf A) \le \inf \varphi(A). \tag{1}$$

There exists a sequence $(v_n)_n \subseteq A$ such that $\lim_{n\to\infty} v_n = \inf A$. Since $\inf \varphi(A) \le \varphi(v_n)$ for all $n \in \mathbb{N}$ and φ is upper semicontinuous, we have

$$\inf \varphi(A) \le \limsup_{n\to\infty} \varphi(v_n) \le \varphi(\inf A),$$

so

$$\inf \varphi(A) \le \varphi(\inf A). \tag{2}$$

By (1) and (2), we deduce that $\varphi(\inf A) = \inf \varphi(A)$ for any bounded $A \subseteq \mathbb{R}$.

Now, let $D, G \in P_{cp}(X)$ and $x \in D$. On account of the above remark, we have

$$\inf_{y\in G} \varphi(d(x,y)) = \varphi(\inf_{y\in G} d(x,y)) = \varphi(d(x,G)) \le \varphi(\sup_{x\in D} d(x,G)) \le \varphi(h(D,G)),$$

which implies that

$$\inf_{y\in G} \rho(f(x), f(y)) \le \inf_{y\in G} \varphi(d(x,y)) \le \varphi(h(D,G)),$$

so

$$\sup_{x\in D} \inf_{y\in G} \rho(f(x), f(y)) \le \varphi(h(D,G)).$$

By a similar argument, we can prove that

$$\sup_{x\in G}\inf_{y\in D}\rho(f(x),f(y))\leq\varphi(h(D,G)),$$

thus

$$h(f(D),f(G))=\max\left\{\sup_{x\in D}\inf_{y\in G}\rho(f(x),f(y)),\sup_{x\in G}\inf_{y\in D}\rho(f(x),f(y))\right\}\leq\varphi(h(D,G)).$$

□

Lemma 4. *If (X,d,s) is a b-metric space, then*

$$h(D_1\times\ldots\times D_m,G_1\times\ldots\times G_m)\leq\max\{h(D_1,G_1),\ldots,h(D_m,G_m)\},$$

for all $D_1,\ldots,D_m,G_1,\ldots,G_m\in P_{cp}(X)$.

Proof. The justification of this claim is based on the definition of the Hausdorff–Pompeiu b-metric and the following two facts:

$$\inf_{x_1\in X_1,\ldots,x_m\in X_m}\max\{x_1,\ldots,x_m\}=\max\{\inf X_1,\ldots,\inf X_m\}\tag{3}$$

and

$$\sup_{x_1\in X_1,\ldots,x_m\in X_m}\max\{x_1,\ldots,x_m\}=\max\{\sup X_1,\ldots,\sup X_m\}\tag{4}$$

for any bounded subsets $X_1,\ldots,X_m\subseteq\mathbb{R}$.

Let us prove (3). One can prove (4) in much the same way. Fix $x_1\in X_1,\ldots,x_m\in X_m$. Then

$$\inf X_i\leq x_i\leq\max\{x_1,\ldots,x_m\},$$

for all $i\in\{1,\ldots,m\}$, hence $\max\{\inf X_1,\ldots,\inf X_m\}\leq\max\{x_1,\ldots,x_m\}$, so

$$\max\{\inf X_1,\ldots,\inf X_m\}\leq\inf_{x_1\in X_1,\ldots,x_m\in X_m}\max\{x_1,\ldots,x_m\}.$$

If it were true that

$$\max\{\inf X_1,\ldots,\inf X_m\}<\inf_{x_1\in X_1,\ldots,x_m\in X_m}\max\{x_1,\ldots,x_m\},$$

there would be $\rho>0$ such that

$$\max\{\inf X_1,\ldots,\inf X_m\}<\rho<\inf_{x_1\in X_1,\ldots,x_m\in X_m}\max\{x_1,\ldots,x_m\}.$$

This means that $\inf X_1<\rho,\ldots,\inf X_m<\rho$, so we find $x_1^0\in X_1,\ldots,x_m^0\in X_m$ satisfying $x_1^0\leq\rho,\ldots,x_m^0\leq\rho$, and so $\max\{x_1^0,\ldots,x_m^0\}\leq\rho$.
This implies that

$$\rho<\inf_{x_1\in X_1,\ldots,x_m\in X_m}\max\{x_1,\ldots,x_m\}\leq\max\left\{x_1^0,\ldots,x_m^0\right\}\leq\rho,$$

which is impossible. Therefore, (3) must be true.

We now proceed to show that

$$h(D_1\times\ldots\times D_m,G_1\times\ldots\times G_m)\leq\max\{h(D_1,G_1),\ldots,h(D_m,G_m)\}$$

for all $D_1, \ldots, D_m, G_1, \ldots, G_m \in P_{cp}(X)$. Observe that if $x_1 \in D_1, \ldots, x_m \in D_m$, then

$$\inf_{y_1 \in G_1, \ldots, y_m \in G_m} \max\{d(x_1, y_1), \ldots, d(x_m, y_m)\} \overset{(3)}{=} \max\left\{\inf_{y_1 \in G_1} d(x_1, y_1), \ldots, \inf_{y_m \in G_m} d(x_m, y_m)\right\}$$
$$= \max\{d(x_1, G_1), \ldots, d(x_m, G_m)\} \leq \max\{h(D_1, G_1), \ldots, h(D_m, G_m)\},$$

which leads to

$$\sup_{x_1 \in D_1, \ldots, x_m \in D_m} \inf_{y_1 \in G_1, \ldots, y_m \in G_m} \max\{d(x_1, y_1), \ldots, d(x_m, y_m)\} \leq \max\{h(D_1, G_1), \ldots, h(D_m, G_m)\},$$

implying that

$$\sup_{x_1 \in D_1, \ldots, x_m \in D_m} d_{max}((x_1, \ldots, x_m), G_1 \times \ldots \times G_m) \leq \max\{h(D_1, G_1), \ldots, h(D_m, G_m)\}.$$

The same reasoning can be used to conclude that

$$\sup_{y_1 \in G_1, \ldots, y_m \in G_m} d_{max}((y_1, \ldots, y_m), D_1 \times \ldots \times D_m) \leq \max\{h(D_1, G_1), \ldots, h(D_m, G_m)\},$$

and the proof is complete. □

Corollary 1. *Let (X, d, s) be a b-metric space and $S = ((X, d, s), \{f_1, \ldots, f_n\})$ a GIFS of order m on X. If each $f_i : X^m \to X$, $i \in \{1, \ldots, n\}$ is a φ-contraction, then F_S is a φ-contraction (with the same φ).*

Proof. We have

$$\begin{aligned} h(F_S(D_1, \ldots, D_m), F_S(G_1, \ldots, G_m)) &= h\left(\bigcup_{k=1}^{n} f_k(D_1 \times \ldots \times D_m), \bigcup_{k=1}^{n} f_k(G_1 \times \ldots \times G_m)\right) \\ &\leq \sup_{k \in \{1, \ldots, n\}} h(f_k(D_1 \times \ldots \times D_m), f_k(G_1 \times \ldots \times G_m)) \\ &\leq \varphi(h(D_1 \times \ldots \times D_m, G_1 \times \ldots \times G_m)) \\ &\leq \varphi(\max\{h(D_1, G_1), \ldots, h(D_m, G_m)\}), \end{aligned}$$

for all $D_1, \ldots, D_m, G_1, \ldots, G_m \in P_{cp}(X)$. □

Combining Theorem 4 and Corollary 1, we obtain the following result.

Theorem 5. *If (X, d, s) is a complete b-metric space and $S = ((X, d, s), \{f_1, \ldots, f_n\})$ is a GIFS of order m on X, then there exists a unique $A_S \in P_{cp}(X)$ such that $F_S(A_S, \ldots, A_S) = A_S$. Moreover, for any $D_1, \ldots, D_m \in P_{cp}(X)$, the following sequence of iterates*

$$D_{k+m} = F_S(D_k, \ldots, D_{k+m-1}), k \in \mathbb{N}$$

converges to A_S with respect to the Hausdorff–Pompeiu b-metric h.

3.2. The Continuous Dependence of the Attractor on Parameters

In this subsection, we present a theorem concerning the continuous dependence of the attractor of a GIFS on parameters.

Theorem 6 ([18])**.** *Let (X, d, s) be a b-metric space, (K, d_K) be a compact metric space and denote by $C(K, X)$ the family of all continuous functions from K to X. Then*

(i) $(C(K, X), d_\infty, s)$ *is a b-metric space, where*

$$d_\infty(f, g) = \sup_{x \in K} d(f(x), g(x)),$$

for all $f, g \in C(K, X)$.

(ii) *If (X, d, s) is complete and d is lower semicontinuous, then $(C(K, X), d_\infty, s)$ is complete.*

We omit the proof of the following lemma since it is identical to that from the metric case.

Lemma 5. *Let (X, d, s) and (Y, ρ, r) be b-metric spaces and $f : X \to Y$. Then, f is continuous if and only if $f_{|A}$ is continuous for any compact subset $A \subseteq X$.*

Following the steps of Theorem 3.2 in [19], we prove the following result.

Proposition 3. *Let (X, d, s) be a complete b-metric space and suppose that d is lower semicontinuous. If $f_n : X \to X$, $n \in \mathbb{N}$ is a sequence of φ-contractions (with the same φ) that converges pointwise to a function $f : X \to X$, then f is a φ-contraction (with the same φ), and the sequence of fixed points of f_n converges to the fixed point of f.*

Proof. We have $d(f_n(x), f_n(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$ and $n \in \mathbb{N}$, so

$$\liminf_{n\to\infty} d(f_n(x), f_n(y)) \leq \varphi(d(x, y)). \tag{5}$$

Since d is lower semicontinuous and

$$d(f_n(x), f(x)) \underset{n}{\to} 0, d(f_n(y), f(y)) \underset{n}{\to} 0,$$

we can assert that

$$d(f(x), f(y)) \leq \liminf_{n\to\infty} d(f_n(x), f_n(y)) \overset{(5)}{\leq} \varphi(d(x, y)).$$

Let α be the unique fixed point of f and α_n be the unique fixed point of f_n, where $n \in \mathbb{N}$. We want to prove that $\lim_{n\to\infty} \alpha_n = \alpha$.

Set $\Lambda = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$. Note that Λ endowed with the usual distance becomes a metric space. Define $F : \Lambda \times X \to X$ by

$$\begin{cases} F\left(\frac{1}{n}, x\right) = f_n(x) \\ F(0, x) = f(x) \end{cases}$$

for all $n \in \mathbb{N}, x \in X$. Note that $\lim_{n\to\infty} F\left(\frac{1}{n}, x\right) = F(0, x)$ for all $x \in X$. Let $l : \Lambda \to X$ be defined by

$$\begin{cases} l\left(\frac{1}{n}\right) = \alpha_n \\ l(0) = \alpha \end{cases}$$

for all $n \in \mathbb{N}$. We shall have completed the proof if we prove that l is continuous, because

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} l\left(\frac{1}{n}\right) = l(0) = \alpha.$$

Fix a nonempty compact subset $\Omega \subseteq \Lambda$. By Lemma 5, it suffices to prove that l is continuous on Ω. Let $Tg : \Omega \to X$ be defined by $(Tg)(\lambda) = F(\lambda, g(\lambda))$ for all $\lambda \in \Omega$ and

$g \in C(\Omega, X)$. Since F is continuous in the first variable and nonexpansive in the second variable, F is continuous. Indeed, this follows from

$$\begin{aligned} d(F(\lambda_1, x_1), F(\lambda_2, x_2)) &\leq s[d(F(\lambda_1, x_1), F(\lambda_2, x_1)) + d(F(\lambda_2, x_1), F(\lambda_2, x_2))] \\ &= s\left[d(f_{\frac{1}{\lambda_1}}(x_1), f_{\frac{1}{\lambda_2}}(x_1)) + d(f_{\frac{1}{\lambda_2}}(x_1), f_{\frac{1}{\lambda_2}}(x_2))\right] \\ &\leq s\left[d(f_{\frac{1}{\lambda_1}}(x_1), f_{\frac{1}{\lambda_2}}(x_1)) + \varphi(d(x_1, x_2))\right], \end{aligned}$$

for all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1, x_2 \in X$. Consequently, $Tg \in C(\Omega, X)$ for any $g \in C(\Omega, X)$.

Now, for $g, h \in C(\Omega, X)$ and $\lambda \in \Omega$, we have

$$d((Tg)(\lambda), (Th)(\lambda)) = d(F(\lambda, g(\lambda)), F(\lambda, h(\lambda))) \leq \varphi(d(g(\lambda), h(\lambda))) \leq \varphi(d_\infty(g, h)),$$

thus $d_\infty(Tg, Th) \leq \varphi(d_\infty(g, h))$, i.e., T is a φ-contraction. By Theorem 6, we know that $(C(\Omega, X), d_\infty, s)$ is complete. Accordingly, there exists a unique $g^* \in C(\Omega, X)$ such that

$$g^*(\lambda) = (Tg^*)(\lambda) = \begin{cases} F(\lambda, g^*(\lambda)) = f_{\frac{1}{\lambda}}(g^*(\lambda)), \ \lambda \neq 0 \\ F(0, g^*(0)) = f(g^*(0)), \ \lambda = 0 \end{cases},$$

thus

$$g^*(\lambda) = \begin{cases} \alpha_{\frac{1}{\lambda}} = l(\lambda), \ \lambda \neq 0 \\ \alpha = l(0), \ \lambda = 0 \end{cases},$$

for any $\lambda \in \Omega$, which implies that $g^* = l_{|\Omega}$. Since g^* is continuous, we infer that $l_{|\Omega}$ is continuous, hence, in view of Lemma 5, l is continuous. □

Corollary 2. *Let (X, d, s) be a complete b-metric space and suppose that d is lower semicontinuous. If $f_n : X^m \to X$, $n \in \mathbb{N}$ is a sequence of φ-contractions (with the same φ) that converges pointwise to $f : X^m \to X$, then f is also a φ-contraction (with the same φ) and the sequence of fixed points of f_n converges to the fixed point of f.*

Proof. Since f_n is a φ-contraction, for every $n \in \mathbb{N}$, we have that

$$d(f_n(\overline{x}), f_n(\overline{y})) \leq \varphi(d_{max}(\overline{x}, \overline{y})),$$

hence

$$\liminf_{n\to\infty} d(f_n(\overline{x}), f_n(\overline{y})) \leq \varphi(d_{max}(\overline{x}, \overline{y})), \tag{6}$$

for all $\overline{x}, \overline{y} \in X^m$. Since $d(f_n(\overline{x}), f(\overline{x})) \underset{n}{\to} 0, d(f_n(\overline{y}), f(\overline{y})) \underset{n}{\to} 0$ and d is lower semicontinuous, we have

$$d(f(\overline{x}), f(\overline{y})) \leq \liminf_{n\to\infty} d(f_n(\overline{x}), f_n(\overline{y})) \overset{(6)}{\leq} \varphi(d_{max}(\overline{x}, \overline{y})),$$

for all $\overline{x}, \overline{y} \in X^m$, so f is a φ-contraction.

Define $g_n, g : X \to X$ by $g_n(x) = f_n(x, \ldots, x)$ and $g(x) = f(x, \ldots, x)$ for any $x \in X$ and $n \in \mathbb{N}$. Since g and g_n are φ-contractions, by Theorem 1, we infer that there exist uniquely $\alpha, \alpha_n \in X$ such that $g(\alpha) = \alpha = f(\alpha, \ldots, \alpha)$ and $g_n(\alpha_n) = \alpha_n = f_n(\alpha_n, \ldots, \alpha_n)$ for all $n \in \mathbb{N}$. Now, using Proposition 3, we deduce that $\lim\limits_{n\to\infty} \alpha_n = \alpha$. □

Lemma 6. *Let (X, d, s) and (Y, ρ, s) be two complete b-metric spaces such that ρ is lower semicontinuous. If $f_n : X \to Y$, $n \in \mathbb{N}$ is a sequence of φ-contractions (with the same φ) that converges pointwise to $f : X \to Y$ on a dense subset of X, then $(f_n)_n$ converges uniformly on compact sets to f.*

Proof. As in the proof of Proposition 3, we obtain that f is a φ-contraction, so

$$\rho(f(x), f(y)) \leq \varphi(d(x,y)) < d(x,y)$$

for all $x, y \in X$.

Set $A = \left\{x \in X : f_n(x) \underset{n}{\to} f(x)\right\}$ and note that $\overline{A} = X$. Let $K \subseteq X$ be a compact set and $\varepsilon > 0$. Since K is compact, by Theorem 1, we know that there exist $p \in \mathbb{N}$ and $x_1, \ldots, x_p \in K$ such that $K \subseteq \bigcup_{i=1}^{p} B\left(x_i, \frac{\varepsilon}{4s^2(s+1)}\right)$.

Since $\overline{A} = X$, there exist $y_1, \ldots, y_p \in A$ with $y_i \in B\left(x_i, \frac{\varepsilon}{4s^2(s+1)}\right)$ for all $i \in \{1, \ldots, p\}$.

Since $\lim_{n\to\infty} \rho(f_n(y_i), f(y_i)) = 0$ for all $i \in \{1, \ldots, p\}$, we can find $n_\varepsilon \in \mathbb{N}$ such that $\rho(f_n(y_i), f(y_i)) < \frac{\varepsilon}{2s^2}$ for all $i \in \{1, \ldots, p\}$, if $n \geq n_\varepsilon$.

Let $x \in K$. Choose $i \in \{1, \ldots, p\}$ such that $x \in B\left(x_i, \frac{\varepsilon}{4s^2(s+1)}\right)$. For any $n \geq n_\varepsilon$, we have

$$\begin{aligned}
\rho(f_n(x), f(x)) &\leq s\rho(f_n(x), f_n(y_i)) + s^2\rho(f_n(y_i), f(y_i)) + s^2\rho(f(y_i), f(x)) \\
&\leq (s+s^2)d(x, y_i) + s^2\frac{\varepsilon}{2s^2} \\
&\leq (s+s^2)[sd(x, x_i) + sd(x_i, y_i)] + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + s^2(s+1)\left(\frac{\varepsilon}{4s^2(s+1)} + \frac{\varepsilon}{4s^2(s+1)}\right) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,
\end{aligned}$$

which means that $(f_n)_n$ converges uniformly to f on K. □

Lemma 7. *Let (X, d, s) be a complete b-metric space and suppose that d is lower semicontinuous. If $f, f_n : X^m \to X$, $n \in \mathbb{N}$ are φ-contractions (with the same φ) such that $(f_n)_n$ converges uniformly to f on compact sets, then*

$$h(f_n(D_1 \times \ldots \times D_m), f(D_1 \times \ldots \times D_m)) \underset{n}{\to} 0$$

for any $D_1, \ldots, D_m \in P_{cp}(X)$.

Proof. We have

$$\begin{aligned}
&h(f_n(D_1 \times \ldots \times D_m), f(D_1 \times \ldots \times D_m)) = \\
&= h\left(\bigcup_{x_1 \in D_1} \ldots \bigcup_{x_m \in D_m} \{f_n(x_1, \ldots, x_m)\}, \bigcup_{x_1 \in D_1} \ldots \bigcup_{x_m \in D_m} \{f(x_1, \ldots, x_m)\}\right) \\
&\leq \sup_{x_1 \in D_1, \ldots, x_m \in D_m} d(f_n(x_1, \ldots, x_m), f(x_1, \ldots, x_m)) \underset{n}{\to} 0
\end{aligned}$$

for all $D_1, \ldots, D_m \in P_{cp}(X)$. □

Lemma 8. *Let (X, d, s) be a complete b-metric space such that d is lower semicontinuous. Then, the Hausdorff–Pompeiu b-metric h is also lower semicontinuous.*

Proof. Let $(D_n)_n, (G_n)_n \subseteq P_{cp}(X)$ and $D, G \in P_{cp}(X)$ such that $h(D_n, D) \underset{n}{\to} 0$ and $h(G_n, G) \underset{n}{\to} 0$. We will prove that

$$h(D, G) \leq \liminf_{n\to\infty} h(D_n, G_n) = L.$$

We begin by proving that $d(x,G) \leq L$ for any $x \in D$. By the definition of the Hausdorff–Pompeiu b-metric, for any $n \in \mathbb{N}$, there exists $x_n \in D_n$ such that

$$d(x,x_n) < h(D,D_n) + \frac{1}{n},$$

so $\lim\limits_{n\to\infty} d(x,x_n) = 0$. For any $n \in \mathbb{N}$ we can find $y_n \in G_n$ such that

$$d(x_n,y_n) < h(D_n,G_n) + \frac{1}{n}.$$

Since $h(D_n,D) \underset{n}{\to} 0$ and $h(G_n,G) \underset{n}{\to} 0$, by the s-relaxed triangle inequality, we deduce that the sequence $(h(D_n,G_n))_n$ is bounded, so there exists $(n_k)_k \subseteq \mathbb{N}$ such that $\lim\limits_{k\to\infty} h(D_{n_k},G_{n_k}) = L$. From the above, it follows that

$$d(x_{n_k},y_{n_k}) < h(D_{n_k},G_{n_k}) + \frac{1}{n_k}$$

for all $k \in \mathbb{N}$.

We can choose $z_{n_k} \in G$, with $k \in \mathbb{N}$ such that

$$d(y_{n_k},z_{n_k}) < h(G_{n_k},G) + \frac{1}{k}.$$

Since $G \in P_{cp}(X)$, there exists a subsequence $(z_{n_{k_p}})_p \subseteq (z_{n_k})_k$ and $z \in G$ such that

$$\lim_{p\to\infty} d(z_{n_{k_p}},z) = 0.$$

Then, from

$$d(y_{n_{k_p}},z) \leq sd(y_{n_{k_p}},z_{n_{k_p}}) + sd(z_{n_{k_p}},z) < sh(G_{n_{k_p}},G) + s\frac{1}{k_p} + sd(z_{n_{k_p}},z),$$

we deduce that

$$\lim_{p\to\infty} d(y_{n_{k_p}},z) = 0.$$

Now, using the fact that d is lower semicontinuous, it follows that

$$d(x,z) \leq \liminf_{p\to\infty} d(x_{n_{k_p}},y_{n_{k_p}}) \leq L,$$

so

$$d(x,G) \leq d(x,z) \leq \liminf_{n\to\infty} h(D_n,G_n),$$

leading to $\sup\limits_{x\in D} d(x,G) \leq \liminf\limits_{n\to\infty} h(D_n,G_n)$.

In the same manner, one can prove that $\sup\limits_{y\in G} d(y,D) \leq \liminf\limits_{n\to\infty} h(D_n,G_n)$; hence, $h(D,G) \leq \liminf\limits_{n\to\infty} h(D_n,G_n)$, which is our claim. □

Theorem 7. *Let (X,d,s) be a complete b-metric space such that d is lower semicontinuous. If $S = ((X,d,s),\{f_1,\dots,f_n\})$ and $S_k = \left((X,d,s),\left\{f_1^k,\dots,f_n^k\right\}\right)$, $k \in \mathbb{N}$, are GIFSs of order m on X such that $(f_i^k)_k$ converges pointwise to f_i on a dense subset of X^m for any $i \in \{1,\dots,n\}$, then,*

$$A_{S_k} \underset{k}{\to} A_S,$$

with respect to the Hausdorff–Pompeiu b-metric h.

Proof. Combining Lemmas 6–8, we obtain

$$h(F_{S_k}(D_1,\dots,D_m),F_S(D_1,\dots,D_m)) =$$
$$= h\left(\bigcup_{j=1}^{n} f_j^k(D_1\times\dots\times D_m),\bigcup_{j=1}^{n} f_j(D_1\times\dots\times D_m)\right)$$
$$\le \sup_{j\in\{1,\dots,n\}} h(f_j^k(D_1\times\dots\times D_m),f_j(D_1\times\dots\times D_m)) \underset{k}{\to} 0,$$

for all $D_1,\dots,D_m \in P_{cp}(X)$. By Corollary 2, we deduce that $A_{S_k} \underset{k}{\to} A_S$. □

4. Examples

In this section, we present two examples that illustrate Theorem 5.

Example 2. *Consider the complete b-metric space* $([0,1],d,2)$*, where* $d:[0,1]\times[0,1]\to\mathbb{R}_+$*,* $d(x,y)=(x-y)^2$ *for all* $x,y\in[0,1]$ *and the Browder comparison function* $\varphi:\mathbb{R}_+\to\mathbb{R}_+$*,* $\varphi(t)=\frac{t}{4}$ *for all* $t>0$*. Let* $f_1,f_2:[0,1]^2\to[0,1]$ *be given by*

$$f_1(x,y)=\frac{x}{2}, f_2(x,y)=\frac{y}{2}$$

for all $x,y\in[0,1]$*.*

Then

$$d(f_1(x,y),f_1(z,t))=d\left(\frac{x}{2},\frac{z}{2}\right)=\frac{1}{4}(x-z)^2\le\frac{1}{4}\max\left\{(x-z)^2,(y-t)^2\right\}$$
$$=\frac{1}{4}d_{max}((x,y),(z,t)),$$

for all $x,y,z,t\in[0,1]$*, which means that* f_1 *is a* φ*-contraction. Likewise, one can prove that* f_2 *is a* φ*-contraction.*

The set function associated with $S=(([0,1],d,2),\{f_1,f_2\})$ *is* $F_S:P_{cp}([0,1])^2\to P_{cp}([0,1])$ *defined by*

$$F_S(D_1,D_2)=f_1(D_1\times D_2)\cup f_2(D_1\times D_2)=\frac{1}{2}D_1\cup\frac{1}{2}D_2,$$

for all $D_1,D_2\in P_{cp}([0,1])$*. Let us note that* $F_S(\{0\},\{0\})=\{0\}$*. Since the conditions of Theorem 5 are fulfilled, we conclude that* $A_S=\{0\}$*.*

Example 3. *Consider the complete b-metric space* $([0,1],d,2)$*, where* $d:[0,1]\times[0,1]\to\mathbb{R}_+$*,* $d(x,y)=(x-y)^2$ *for all* $x,y\in[0,1]$ *and the Browder comparison function* $\varphi:\mathbb{R}_+\to\mathbb{R}_+$*,* $\varphi(t)=\frac{t}{t+1}$ *for all* $t>0$*. Let* $f_1,f_2:[0,1]^2\to[0,1]$ *be given by*

$$f_1(x,y)=\frac{1}{4}(x+y), f_2(x,y)=\frac{1}{4}(x+y)+\frac{1}{2}$$

for all $x,y\in[0,1]$*. Then*

$$d(f_1(x,y),f_1(z,t))=d\left(\frac{1}{4}(x+y),\frac{1}{4}(z+t)\right)=\left(\frac{1}{4}(x+y)-\frac{1}{4}(z+t)\right)^2$$
$$=\frac{1}{16}[(x-z)+(y-t)]^2,$$

for all $x,y,z,t\in[0,1]$*.*

Suppose that

$$d_{max}((x,y),(z,t))=\max\left\{(x-z)^2,(y-t)^2\right\}=(x-z)^2>0.$$

Note that

$$\frac{1}{16}[(x-z)+(y-t)]^2 \leq \frac{1}{16}4(x-z)^2 = \frac{1}{4}(x-z)^2$$

and

$$\frac{1}{4}(x-z)^2 \leq \frac{(x-z)^2}{(x-z)^2+1} \Leftrightarrow (x-z)^2+1 \leq 4,$$

which is true since $x, z \in [0,1]$. Thus, f_1 is a φ-contraction. Likewise, one can prove that f_2 is a φ-contraction.

The set function associated with $S = (([0,1], d, 2), \{f_1, f_2\})$ is $F_S : P_{cp}([0,1])^2 \to P_{cp}([0,1])$ defined by

$$F_S(D_1, D_2) = f_1(D_1 \times D_2) \cup f_2(D_1 \times D_2) = \frac{1}{4}(D_1+D_2) \cup \left(\frac{1}{4}(D_1+D_2)+\frac{1}{2}\right),$$

for all $D_1, D_2 \in P_{cp}([0,1])$. Let us note that

$$F_S([0,1],[0,1]) = \frac{1}{4}[0,2] \cup \left(\frac{1}{4}[0,2]+\frac{1}{2}\right) = \left[0,\frac{1}{2}\right] \cup \left(\left[0,\frac{1}{2}\right]+\frac{1}{2}\right)$$
$$= \left[0,\frac{1}{2}\right] \cup \left[\frac{1}{2},1\right] = [0,1].$$

Since the conditions of Theorem 5 are fulfilled, we conclude that $A_S = [0,1]$.

5. Discussion

In this paper, we studied generalized iterated function systems (GIFSs) in the context of b-metric spaces. Our results fall within the current effort of extending the classical theory of Hutchinson–Barnsley fractals. We proved that in this framework, a GIFS admits a unique attractor and, if the b-metric is lower semicontinuous, the attractor of the GIFS is a continuous function of the parameters of the GIFS. A possible future direction of research is the problem of finding the Hutchinson measure for a GIFS with probabilities in this setting of b-metric spaces.

Since the classes of Matkowski and Boyd–Wong φ-contractions are incomparable, as one of the reviewers noted, the following open question arises naturally: is the proposed method valid for Boyd–Wong φ-contractions rather than Matkowski contractions?

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