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Results in Mathematics

The Canonical Projection Associated with a Mixed Possibly Infinite Iterated Function System

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Abstract. This paper provides an alternative description for the fixed points of the fractal operator associated with a mixed possibly infinite iterated function system via a canonical projection type function. Some visual aspects of our results are presented.

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1. Introduction

An iterated function system is a pair $((X,d),(f_i)_{i\in I}) = \mathcal{S}$, where (X,d) is a complete metric space, I is a finite set and $f_i : X \to X$, $i \in I$, are Banach contractions. With such a system we can associate the fractal operator $F_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, given by $F_{\mathcal{S}}(K) = \underset{i\in I}{\cup} f_i(K)$, for all $K \in P_{cp}(X) = \{C \mid C \text{ is a non empty compact subset of } X\}$. Moreover, if $\mathcal{S}$ is endowed with probabilities, we can also consider the Markov operator acting on a certain set of probability Borel measures. Via the Banach contraction principle, J. Hutchinson (see [5]) proved that $F_{\mathcal{S}}$ has a unique fixed point $A_{\mathcal{S}}$ (which is called the attractor of $\mathcal{S}$) and that the Markov operator also has a unique fixed point which is called the associated Hutchinson measure. The sets that can be represented as attractors of iterated function systems are called Hutchinson-Barnsley fractals.

The canonical projection associated with $\mathcal{S}$ is a continuous surjection π from the code space $\Lambda(I)$ onto the attractor $A_{\mathcal{S}}$. It is also called "the coding map" (see [12] and [13]), "the address map" (see [9]) or "the coordinate map" (see [8]).

The importance of this projection is emphasized by the following facts:

a) It provides, via the formula $\pi(\Lambda(I)) = A_{\mathcal{S}}$, an alternative description of the attractor of $\mathcal{S}$ and, consequently, it is an important device in the topological study of Hutchinson-Barnsley fractals.
b) It is involved in the alternative presentation of the Hutchinson measure associated with $\mathcal{S}$ as the push-forward measure of the Bernoulli measure on $\Lambda(I)$ through π.
c) It was a source of inspiration for the concept of topological self-similar set introduced by A. Kameyama (see [7]).

The above mentioned theory (which was initiated by J. Hutchinson and developed by M. Barnsley) has been extended in different directions. Two of them are of special interest from the point of view of the present paper:

i) the direction concentrating on systems involving not necessarily finite families of functions (see [2–4, 10, 12, 14] and [15]);
ii) the direction focusing on systems involving functions from larger classes of contractions (see [11] which, together with [6], is an excellent survey on iterated function systems, and the references therein).

We emphasize that the corresponding fractal operator is a Picard operator for all the above mentioned generalizations of the concept of iterated function system.

In the recent work [1], we integrated the previous directions by considering possibly infinite iterated function systems enriched with orbital possibly infinite iterated function systems (called mixed possibly infinite iterated function systems) and we proved that the fractal operator associated with such a system is weakly Picard. Its fixed points are called attractors.

The present paper develops a canonical projection type theory for mixed possibly infinite iterated function systems in order to obtain an alternative description for the attractors. Finally some visual aspects concerning our results are presented.

2. Preliminaries

The generalized pigeonhole principle

In the sequel we will use the following form of the generalized pigeonhole principle: If N objects are placed in k boxes, then at least one box contains at least $[\frac{N-1}{k}] + 1$ objects.

Basic notation

By $\mathbb{N}$ we mean the set $\{1, 2, ...\}$.

For a function $f : X \to X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we mean the composition of f by itself n times and by $f^{[0]}$ we mean Id_X.

For a Lipschitz function $f : X \to Y$, where (X,d) and (Y,ρ) are metric spaces, by $lip(f)$ we denote the Lipschitz constant of f.

For a metric space (X,d) and $A, B \subseteq X$ we shall use the following notation:

- $\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is bounded}\} \overset{not}{=} P_b(X)$
- $\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is closed and bounded}\} \overset{not}{=} P_{b,cl}(X)$
- $\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\} \overset{not}{=} P_{cp}(X)$
- $\inf\limits_{a\in A} d(x,a) \overset{not}{=} d(x,A)$
- $\sup\limits_{a\in A} d(a,B) \overset{not}{=} D(A,B)$
- $\sup\limits_{x,y\in A} d(x,y) \overset{not}{=} diam(A)$.

The code space

Given a set I and $n \in \mathbb{N}$, we consider:

-
$$I^{\mathbb{N}} \overset{not}{=} \Lambda(I)$$

-
$$I^{\{1,\dots,n\}} \overset{not}{=} \Lambda_n(I).$$

So:

- the elements of $\Lambda(I)$, which is called a code space, are written as infinite words $\alpha = \alpha_1\alpha_2...\alpha_n\alpha_{n+1}...$ with letters from I

- the elements of $\Lambda_n(I)$ are written as words $\alpha = \alpha_1\alpha_2...\alpha_n$ having n letters from I and n, which is denoted by $|\alpha|$, is called the length of α.

In the sequel we will use the following notation:

$$\bigcup_{n\in\mathbb{N}\cup\{0\}} \Lambda_n(I) \overset{not}{=} \Lambda^*(I),$$

where $\Lambda_0(I) = \{\lambda\}$ and λ designates the empty word.

If $\alpha = \alpha_1\alpha_2...\alpha_n\alpha_{n+1}.... \in \Lambda(I)$ or if $\alpha = \alpha_1\alpha_2...\alpha_n \in \Lambda_n(B)$ and $m, n \in \mathbb{N}$, $n \geq m$, then we will use the following notation:

$$\alpha_1\alpha_2...\alpha_m \overset{not}{=} [\alpha]_m.$$

By the concatenation of the words α and β, where $\alpha = \alpha_1\alpha_2...\alpha_n \in \Lambda_n(I)$ and $\beta = \beta_1\beta_2...\beta_m\beta_{m+1}... \in \Lambda(I)$, we mean the infinite word $\alpha_1...\alpha_n\beta_1...\beta_m\beta_{m+1}...$ which is denoted by $\alpha\beta$.

For $i \in I$, we introduce the function $\tau_i : \Lambda(I) \to \Lambda(I)$, given by

$$\tau_i(\alpha) = i\alpha,$$

for all $\alpha \in \Lambda(I)$.

$\Lambda(I)$ can be endowed with the metric $d_{\Lambda(I)}$, called the Baire metric, described by:

$$d_{\Lambda}(\alpha,\beta)=\begin{cases}0, & \text{if } \alpha=\beta\\ \frac{1}{2^{\min\{k\in\mathbb{N}\mid \alpha_k\neq\beta_k\}}}, & \text{if } \alpha\neq\beta\end{cases},$$

for all $\alpha=\alpha_1\alpha_2...\alpha_n\alpha_{n+1}...\in\Lambda(I)$ and $\beta=\beta_1\beta_2...\beta_n\beta_{n+1}...\in\Lambda(I)$.

Given $f_i : X\to X$, $i\in I$, and $\alpha=\alpha_1\alpha_2...\alpha_n\in\Lambda_n(I)$, the following notation will be used in the sequel:

$$f_{\alpha_1}\circ f_{\alpha_2}\circ ....\circ f_{\alpha_n}\overset{not}{=}f_\alpha$$

and

$$Id_X\overset{not}{=}f_\lambda.$$

In particular if $I=\{i\}$, we have $f_{\underbrace{i...i}_{n\text{ times}}}=f_i^{[n]}$ for every $n\in\mathbb{N}$.

The Hausdorff-Pompeiu metric

Remark 2.1. i) Given a metric space (X,d) and A and B subsets of X, we have

$$D(A,B)=D(\overline{A},\overline{B}).$$

ii) Given a metric space (X,d) and a set I , we have

$$D(\underset{i\in I}{\cup}A_i,\underset{i\in I}{\cup}B_i)\leq\underset{i\in I}{\sup}D(A_i,B_i),$$

for every A_i and B_i subsets of X .

Proposition 2.2. *Given a metric space (X,d), the function $h^*: P_b(X)\times P_b(X)\to[0,\infty)$, described by*

$$h^*(A,B)=\max\{D(A,B),D(B,A)\},$$

for every $A,B\in P_b(X)$, has the following properties:

i)

$$h^*(\{x\},\{y\})=d(x,y),$$

for every $x,y\in X$;

ii)

$$h^*(A,B)=h^*(\overline{A},\overline{B}),$$

for every $A,B\in P_b(X)$;

iii)

$$h^*(\underset{i\in I}{\cup}A_i,\underset{i\in I}{\cup}B_i)\leq\underset{i\in I}{\sup}h^*(A_i,B_i),$$

for all families $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ of elements of $P_b(X)$ such that $\underset{i\in I}{\cup}A_i\in P_b(X)$ and $\underset{i\in I}{\cup}B_i\in P_b(X)$.

Let us recall two well-known results.

Definition 2.3. *Given a metric space (X,d), the restriction of h^* to $P_{b,cl}(X)\times P_{b,cl}(X)$ is a metric, denoted by h, called the Hausdorff-Pompeiu metric.*

Proposition 2.4. *Given a complete metric space* (X,d) *, the metric space* $(P_{b,cl}(X),h)$ *is complete and if* $(A_n)_{n\in\mathbb{N}}\subseteq P_{b,cl}(X)$ *is Cauchy, then*

$$\lim_{n\to\infty}A_n=\Big\{x\in X \mid \text{ there exist a strictly increasing sequence } (n_k)_{k\in\mathbb{N}}\subseteq\mathbb{N} \text{ and } x_{n_k}\in A_{n_k} \text{ for every } k\in\mathbb{N} \text{ such that } \lim_{k\to\infty}x_{n_k}=x\Big\}.$$

Proposition 2.5. *Given a metric space* (X,d) *and* $(A_n)_{n\in\mathbb{N}}\subseteq P_{b,cl}(X)$ *such that* $A_{n+1}\subseteq A_n$ *for every* $n\in\mathbb{N}$*, we have*

$$\lim_{n\to\infty}h\left(A_n,\underset{n\in\mathbb{N}}{\cap}A_n\right)=0.$$

Possibly infinite iterated function systems

Definition 2.6. *A possibly infinite iterated function system (for short IIFS) is a pair* $((X,d),(f_i)_{i\in I})\overset{not}{=}\mathcal{S}$*, where* (X,d) *is a complete metric space and* $f_i:X\to X$*,* $i\in I$*, are such that:*

a) f_i *is continuous for every* $i\in I$*;*

b) the family $(f_i)_{i\in I}$ *is bounded, i.e.*

$$\underset{i\in I}{\cup}f_i(B)\in P_b(X),$$

for every $B\in P_b(X)$*.*

The function $F_{\mathcal{S}}:P_{b,cl}(X)\to P_{b,cl}(X)$ *, given by*

$$F_{\mathcal{S}}(B)=\overline{\underset{i\in I}{\cup}f_i(B)},$$

for every $B\in P_{b,cl}(X)$*, is called the fractal operator associated with* $\mathcal{S}$*.*

Remark 2.7. In the framework of the above definition, if the functions f_i *are Lipschitz, then*

$$lip(F_{\mathcal{S}})\leq\sup_{i\in I}lip(f_i).$$

Proposition 2.8 *(see Lemma 2.6 from [17])***.** *For each IIFS* $\mathcal{S}=((X,d),(f_i)_{i\in I})$*, we have*

$$F_{\mathcal{S}}^{[n]}(B)=\overline{\underset{\alpha\in\Lambda_n(I)}{\cup}f_\alpha(B)},$$

for every $n\in\mathbb{N}$ *and every* $B\in P_{b,cl}(X)$ *.*

Theorem 2.9 *(see [15])***.** *For each IIFS* $\mathcal{S}=(X,(f_i)_{i\in I})$ *such that the functions* f_i *are Lipschitz and* $\sup_{i\in I}lip(f_i)<1$*,* $F_{\mathcal{S}}$ *is a Banach contraction with respect to* h *and its unique fixed point* $A_{\mathcal{S}}$ *is called the attractor of* $\mathcal{S}$*.*

Moreover, we have:

a) For every $\alpha\in\Lambda(I)$*, the set* $\underset{n\in\mathbb{N}}{\cap}\overline{f_{[\alpha]_n}(A_{\mathcal{S}})}$ *has just one element, which is denoted by* a_α*.*

b)

$$\lim_{n\to\infty} f_{[\alpha]_n}(x) = a_\alpha,$$

for every $x \in X$ *and every* $\alpha \in \Lambda(I)$.

c) The continuous (with respect to the Baire metric) function $\pi : \Lambda(I) \to A_{\mathcal{S}}$*, defined by*

$$\pi(\alpha) = a_\alpha,$$

for every $\alpha \in \Lambda(I)$*, is called the canonical projection associated with* $\mathcal{S}$ *and it has the following properties:*

i)

$$\pi \circ \tau_i = f_i \circ \pi,$$

for every $i \in I$*;*

ii)

$$\overline{\pi(\Lambda(I))} = A_{\mathcal{S}}.$$

Given a metric space (X, d), $x \in X$, $B \subseteq X$ and a family of functions $\mathcal{F} = (f_i)_{i\in I}$, where $f_i : X \to X$, we shall use the notation:

$$\overline{\bigcup_{n\in\mathbb{N}\cup\{0\}} \bigcup_{\alpha\in\Lambda_n(I)} f_\alpha(B)} \overset{not}{=} \mathcal{O}_{\mathcal{F}}(B)$$

and

$$\mathcal{O}_{\mathcal{F}}(\{x\}) \overset{not}{=} \mathcal{O}_{\mathcal{F}}(x).$$

In particular, given an IIFS $\mathcal{S} = ((X, d), (f_i)_{i\in I})$ and $B \in P_{b,cl}(X)$, we shall use the notation

$$\mathcal{O}_{(f_i)_{i\in I}}(B) \overset{not}{=} \mathcal{O}_{\mathcal{S}}(B).$$

Definition 2.10. *An orbital possibly infinite iterated function system (for short oIIFS) is an IIFS* $\mathcal{S}$ =$((X, d), (f_i)_{i\in I})$ *such that:*

a) the family $(f_i)_{i\in I}$ *is equi-uniformly continuous on bounded sets, i.e. for every* $B \in P_b(X)$ *and every* $\varepsilon > 0$ *there exists* $\delta_{\varepsilon,B} > 0$ *such that for all* $x, y \in B$ *and* $i \in I$ *the following implication is valid:* $d(x, y) < \delta_{\varepsilon,B} \Rightarrow d(f_i(x), f_i(y)) < \varepsilon$.

b) there exists $a \in [0, 1)$ *such that*

$$d(f_i(y), f_i(z)) \le a d(y, z),$$

for every $i \in I$, $x \in X$ *and* $y, z \in \mathcal{O}_{(f_i)_{i\in I}}(x)$.

Definition 2.11. *A mixed possibly infinite iterated function system (briefly mIIFS) is an IIFS* $\mathcal{S}$ =$((X, d), (f_i)_{i\in I\cup J})$*, where* I *and* J *are disjoint sets, such that:*

a)

$$lip(f_i) \le 1,$$

for every $i \in I \cup J$*;*

b) there exists $a \in [0, 1)$ *having the following properties:*

b 1)

$$lip(f_i) \le a,$$

for every $i \in I$;

b 2)

$$lip(f_{i|\overline{\mathcal{O}_{(f_i)_{i\in J}}(x)}}) \leq a,$$

for every $x \in X$ *and every* $i \in J$.

Proposition 2.12 *(see [1]). For each mIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ *and every* $B \in P_{b,cl}(X)$ *there exists* $A_B \in P_{b,cl}(X)$ *such that*

$$F_{\mathcal{S}}(A_B) = A_B$$

and

$$\lim_{n\to\infty} h(F_{\mathcal{S}}^{[n]}(B), A_B) = 0,$$

i.e. $F_{\mathcal{S}}$ *is weakly Picard.*

Note that, in the framework of the above definition, $((X,d),(f_i)_{i\in I}) \overset{not}{=} \mathcal{S}_I$ and $((X,d),(f_{\omega i\theta})_{i\in I,\omega,\theta\in\Lambda^*(J)}) \overset{not}{=} \mathfrak{S}$ are IIFSs (see Remark 2.9 and Lemma 3.5 from [1]) and $((X,d),(f_i)_{i\in J}) \overset{not}{=} \mathcal{S}_J$ is an oIIFS.

Given an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $x \in X$, $B \in P_{b,cl}(X)$ and $\alpha = \alpha_1\alpha_2...\alpha_n... \in \Lambda(I\cup J)$ or $\alpha = \alpha_1\alpha_2...\alpha_n \in \Lambda^*(I\cup J)$, we shall use the following notation:

-

$$A_{\{x\}} \overset{not}{=} A_x$$

-

$$\mathcal{O}_{\mathcal{S}_J}(B) \overset{not}{=} \mathcal{O}_J(B) \overset{\text{Lemma 3.36 from [16]}}{\in} P_b(X)$$

-

$$\sup_{x\in B} \max\{diam(\{x\}\cup F_{\mathcal{S}}(\{x\})), diam(\mathcal{O}_{\mathfrak{S}}(x)), diam(\mathcal{O}_J(x))\} \overset{not}{=} N_B \in \mathbb{R}$$

-

$$card(\{l \in \mathbb{N} \mid \alpha_l \in I\}) \overset{not}{=} n_I(\alpha).$$

- $\Lambda_1(I\cup J)$ denotes the set of finite words with letters from $I\cup J$ ending with a letter from I

- $\Lambda_2(I\cup J)$ denotes the set of finite words with letters from $I\cup J$ starting with a letter from I

- $\Lambda_3(I\cup J) = \Lambda_1(I\cup J)\cap\Lambda_2(I\cup J)$ denotes the set of finite words with letters from $I\cup J$ starting and ending with letters from I

- $\Sigma_0(I,J)$ denotes the set of finite words with letters from $I\cup J\cup\Lambda(J)$ having the form

$$\beta_0\gamma_1\beta_1...\gamma_n\beta_n,$$

with $n \in \mathbb{N}$, where

$$\beta_0 \in \{\lambda\}\cup\Lambda_1(I\cup J),$$
$$\beta_n \in \{\lambda\}\cup\Lambda_2(I\cup J),$$
$$\gamma_k \in \Lambda(J)$$

for every $k \in \{1, ..., n\}$ and

$$\beta_k \in \Lambda_3(I \cup J),$$

for every $k \in \{1, ..., n-1\}$ if $n \geq 2$

-
$$\Sigma_1(I, J) = \{\alpha \in \Lambda(I \cup J) \mid n_I(\alpha) = \infty\}$$

-
$$\Sigma(I, J) = \Sigma_0(I, J) \cup \Sigma_1(I, J).$$

Additionally, we consider the function $g : X \to P_{b,cl}(X)$, given by

$$g(x) = A_x,$$

for every $x \in X$.

Lemma 2.13. *Given an mIIFS* $\mathcal{S} = ((X, d), (f_i)_{i \in I \cup J})$*, we have*

$$lip(g) \leq 1.$$

Proof. Indeed, we have

$$\begin{aligned} h(g(x), g(y)) &= h(A_x, A_y) \\ &\leq h(A_x, F_{\mathcal{S}}^{[n]}(\{x\})) + h(F_{\mathcal{S}}^{[n]}(\{x\}), F_{\mathcal{S}}^{[n]}(\{y\})) + h(F_{\mathcal{S}}^{[n]}(\{y\}), A_y) \\ &\overset{\text{Remark 2.7 \&Proposition 2.2, i)}}{\leq} h(A_x, F_{\mathcal{S}}^{[n]}(\{x\})) + h(F_{\mathcal{S}}^{[n]}(\{y\}), A_y) + d(x, y), \end{aligned}$$

for all $n \in \mathbb{N}$, so, by passing to the limit as $n \to \infty$, we get

$$h(g(x), g(y)) \leq d(x, y),$$

for all $x, y \in X$. □

Lemma 2.14. *Given an mIIFS* $\mathcal{S} = ((X, d), (f_i)_{i \in I \cup J})$*, we have*

$$A_{f_i(x)} \subseteq A_x,$$

for all $x \in X$ *and* $i \in I \cup J$.

Proof. We have

$$\begin{aligned} A_{f_i(x)} &\overset{\text{Proposition 2.12}}{=} \lim_{n \to \infty} F_{\mathcal{S}}^{[n]}(\{f_i(x)\}) \overset{\text{Proposition 2.4}}{=} \Big\{ y \in X \mid \text{there exist} \\ &\quad \text{a strictly increasing sequence } (n_k)_{k \in \mathbb{N}} \text{ of natural numbers and} \\ &\quad y_{n_k} \in F_{\mathcal{S}}^{[n_k]}(\{f_i(x)\}) \text{ for every } k \in \mathbb{N} \text{ such that } y = \lim_{k \to \infty} y_{n_k} \Big\} \\ &\overset{F_{\mathcal{S}}^{[n_k]}(\{f_i(x)\}) \subseteq F_{\mathcal{S}}^{[n_k+1]}(\{x\})}{\subseteq} \Big\{ y \in X \mid \text{there exist a strictly increasing sequence} \\ &\quad (m_k)_{k \in \mathbb{N}} \text{ of natural numbers and } y_{m_k} \in F_{\mathcal{S}}^{[m_k]}(\{x\}) \text{ for every } k \in \mathbb{N} \\ &\quad \text{such that } y = \lim_{k \to \infty} y_{m_k} \Big\} = A_x, \end{aligned}$$

for all $x \in X$ and $i \in I \cup J$. □

3. The Main Results

Our first result is a counterpart of Theorem 2.9, b).

Proposition 3.1. *For each mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha \in \Lambda(I\cup J)$ *and* $x \in X$, *the sequence* $(f_{[\alpha]_n}(x))_{n\in\mathbb{N}}$ *is convergent.*

Proof. We divide the proof into two cases:

a) $\alpha \in \Sigma_1(I,J)$, i.e. $n_I(\alpha)$ is infinite;

b) $\alpha \in \Lambda(I\cup J)\setminus\Sigma_1(I,J)$, i.e. $n_I(\alpha)$ is finite.

In the first case, we can find a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that

$$\alpha_{n_k} \in I,$$

for every $k \in \mathbb{N}$ and

$$\alpha_n \notin I, \text{ i.e. } \alpha_n \in J,$$

for every $n \in \mathbb{N} \setminus \{n_k \mid k \in \mathbb{N}\}$.

Claim. The sequence $(f_{[\alpha]_n}(x))_{n\in\mathbb{N}}$ is Cauchy.

Justification of the claim. Adopting the notation

$$\alpha_{n_{k-1}+1}\alpha_{n_{k-1}+2}...\alpha_{n_k} \overset{not}{=} \beta_k,$$

we have

$$d(f_{[\alpha]_{n_k}}(x), f_{[\alpha]_{n_{k+1}}}(x)) = d(f_{[\alpha]_{n_k}}(x), f_{[\alpha]_{n_k}\beta_{k+1}}(x)) \overset{\text{Definition 2.11, b 1)}}{\leq}$$
$$\leq a^k d(x, f_{\beta_{k+1}}(x)) \leq a^k diam(O_{\mathfrak{S}}(x)),$$

so

$$d(f_{[\alpha]_{n_k}}(x), f_{[\alpha]_{n_{k+p}}}(x)) \leq (a^k + a^{k+1} + ... + a^{k+p-1}) diam(O_{\mathfrak{S}}(x))$$
$$\leq \frac{a^k}{1-a} diam(O_{\mathfrak{S}}(x)), \tag{1}$$

for every $k, p \in \mathbb{N}$.

For $k, m, n \in \mathbb{N}$ such that $n_k \leq n \leq m$, there exist $s, t \in \mathbb{N}$, $k \leq s \leq t$ having the property that $n_s \leq n < n_{s+1}$ and $n_t \leq m < n_{t+1}$. Hence we get

$$d(f_{[\alpha]_n}(x), f_{[\alpha]_{n_s}}(x)) \leq a^k diam(O_J(x)) \tag{2}$$

and

$$d(f_{[\alpha]_m}(x), f_{[\alpha]_{n_t}}(x)) \leq a^k diam(O_J(x)). \tag{3}$$

Consequently

$$d(f_{[\alpha]_m}(x), f_{[\alpha]_n}(x)) \leq d(f_{[\alpha]_m}(x), f_{[\alpha]_{n_t}}(x)) + d(f_{[\alpha]_{n_s}}(x), f_{[\alpha]_{n_t}}(x))$$
$$+ d(f_{[\alpha]_{n_s}}(x), f_{[\alpha]_n}(x)) \overset{(1),\ (2)\ \&\ (3)}{\leq} 2a^k diam(O_J(x))$$
$$+ \frac{a^k}{1-a} diam(O_{\mathfrak{S}}(x)). \tag{4}$$

The last inequality yields claim's validity.

In view of the Claim we infer that, in the first case, $(f_{[\alpha]_n}(x))_{k\in\mathbb{N}}$ is convergent.

In the second case, there exist $\beta \in \Lambda^*(I \cup J)$ and $\gamma \in \Lambda(J)$ such that $\alpha = \beta\gamma$ and we get

$$(f_{[\alpha]_{|\beta|+n}}(x), f_{[\alpha]_{|\beta|+n+p}}(x)) \overset{\text{Definition 2.11, a)}}{\leq} d(f_{[\gamma]_n}(x), f_{[\gamma]_{n+p}}(x)) \overset{\text{Definition 2.11, b 2)}}{\leq} a^n diam(O_J(x)), \tag{5}$$

for every $n, p \in \mathbb{N}$. The previous inequality ensures that $(f_{[\alpha]_n}(x))_{n\in\mathbb{N}}$ is Cauchy, so it is also convergent.

Now the proof is complete. □

For an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ and $\alpha \in \Lambda(I \cup J)$, based on Proposition 3.1, we can consider the function $a_\alpha : X \to X$, given by

$$a_\alpha(x) = \lim_{n\to\infty} f_{[\alpha]_n}(x),$$

for every $x \in X$.

A closer look at the inequalities (4) and (5) from the proof of the above Proposition leads to the conclusion that the convergence of the sequence $(f_{[\alpha]_n}(x))_{n\in\mathbb{N}}$ is uniform with respect to x in a bounded subset of X. More precisely, we can state the following:

Corollary 3.2. *For each mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha \in \Lambda(I \cup J)$ *and* $B \in P_b(X)$, *we have*

$$\lim_{n\to\infty} \sup_{x\in B} d(f_{[\alpha]_n}(x), a_\alpha(x)) = 0.$$

Proposition 3.3. *For an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha \in \Lambda(I \cup J)$ *and* $B \in P_{b,cl}(X)$, *we have*

$$\overline{a_\alpha(B)} \subseteq A_B.$$

Proof. For each $B \in P_{b,cl}(X)$, $\alpha \in \Lambda(I \cup J)$, $x \in B$ and $n \in \mathbb{N}$, we have

$$f_{[\alpha]_n}(x) \overset{\text{Proposition 2.8}}{\in} F_{\mathcal{S}}^{[n]}(B),$$

so, since $\lim_{n\to\infty} F_{\mathcal{S}}^{[n]}(B) \overset{\text{Proposition 2.12}}{=} A_B$, via Proposition 2.4, we infer that

$$a_\alpha(x) \in A_B.$$

Therefore

$$a_\alpha(B) \subseteq A_B,$$

so

$$\overline{a_\alpha(B)} \subseteq A_B.$$

□

Proposition 3.4. *For each mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ and $\alpha \in \Lambda(I\cup J)$, we have:*

a)

$$lip(a_\alpha) \leq 1;$$

b) a_α is constant for every $\alpha \in \Sigma_1(I,J)$.

Proof. a) Note that

$$d(f_{[\alpha]_n}(x), f_{[\alpha]_n}(y)) \overset{\text{Definition 2.11, a) \& b 1)}}{\leq} a^{n_I([\alpha]_n)} d(x,y) \leq d(x,y), \quad (1)$$

for all $n \in \mathbb{N}$ and $x, y \in X$.

Via (1), by passing to the limit as n goes to ∞, we infer that

$$d(a_\alpha(x), a_\alpha(y)) \leq d(x,y),$$

for all $x, y \in X$, so $lip(a_\alpha) \leq 1$.

b) If $n_I(\alpha) = \infty$, then $\lim\limits_{n\to\infty} n_I([\alpha]_n) = \infty$, and, by passing again to the limit as n goes to ∞ in (1), we conclude that

$$a_\alpha(x) = a_\alpha(y),$$

for all $x, y \in X$, so $a_\alpha$is constant. □

Proposition 3.5. *Given an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, we have*

$$a_{i\alpha} = f_i \circ a_\alpha,$$

for all $i \in I \cup J$ and $\alpha \in \Lambda(I\cup J)$.

Proof. Indeed, we have

$$f_i(a_\alpha(x)) = f_i\left(\lim_{n\to\infty} f_{[\alpha]_n}(x)\right) \overset{f_i \text{ continuous}}{=}$$
$$= \lim_{n\to\infty} f_i(f_{[\alpha]_n}(x)) = \lim_{n\to\infty} f_{i[\alpha]_n}(x) = \lim_{n\to\infty} f_{[i\alpha]_{n+1}}(x) = a_{i\alpha}(x),$$

for all $x \in X$, $i \in I \cup J$ and $\alpha \in \Lambda(I\cup J)$. □

Proposition 3.6. *Given an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, we have*

$$\lim_{n\to\infty} h^*(f_{[\alpha]_n}(B), a_\alpha(B)) = 0,$$

for all $B \in P_b(X)$ and $\alpha \in \Lambda(I\cup J)$.

Proof. Let us consider $B \in P_b(X)$and $\alpha \in \Lambda(I\cup J)$ arbitrarily chosen, but fixed.

First let us note that

$$a_\alpha(B) \in P_b(X)$$

and

$$f_{[\alpha]_n}(B) \in P_b(X),$$

for every $n \in \mathbb{N}$, since $B \in P_b(X)$ and a_α and $f_{[\alpha]_n}$ are Lipschitz.

Finally, we have

$$h^*(f_{[\alpha]_n}(B), a_\alpha(B)) \overset{\text{Proposition 2.2, iii)}}{\leq}$$

$$\leq \sup_{x\in B} h^*(f_{[\alpha]_n}(\{x\}), a_\alpha(\{x\})) \overset{\text{Proposition 2.2, i)}}{=} \sup_{x\in B} d(f_{[\alpha]_n}(x), a_\alpha(x)),$$

for every $n \in \mathbb{N}$, so, using the squeeze theorem, via Corollary 3.2, we get the conclusion. □

Remark 3.7. Given an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *it follows from Claim 2 from the proof of Theorem 3.7 from* [1] *that*

$$\lim_{n\to\infty} \sup_{x\in B} h(F_{\mathcal{S}}^{[n]}(\{x\}), A_x) = 0,$$

for every $B \in P_{b,cl}(X)$.

Proposition 3.8. *Given an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$A_B = \overline{\underset{x\in B}{\cup} A_x},$$

for all $B \in P_{b,cl}(X)$.

Proof. Let us consider $B \in P_{b,cl}(X)$ arbitrarily chosen, but fixed.

In view of Proposition 2.8, we have

$$\underset{x\in B}{\cup} A_x \subseteq A_B,$$

so $\underset{x\in B}{\cup} A_x \in P_b(X)$.

We have

$$h(A_B, \overline{\underset{x\in B}{\cup} A_x}) \leq h(A_B, F_{\mathcal{S}}^{[n]}(B)) + h^*(F_{\mathcal{S}}^{[n]}(B), \underset{x\in B}{\cup} A_x)$$

$$\overset{\text{Proposition 2.2, iii)}}{\leq} h(A_B, F_{\mathcal{S}}^{[n]}(B)) + \sup_{x\in B} h(F_{\mathcal{S}}^{[n]}(\{x\}), A_x),$$

for all $n \in \mathbb{N}$, so using the squeeze theorem, via Remark 3.7, we get the conclusion. □

Proposition 3.9. *Let us consider an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha = \alpha_1\alpha_2 \ldots\alpha_n\ldots \in \Lambda(I\cup J)$ *and* $B \in P_{b,cl}(X)$ *such that* $F_{\mathcal{S}}(B) \subseteq B$.

a) If $\alpha \in \Sigma_1(I,J)$, *i.e.* $n_I(\alpha) = \infty$, *then*

$$\underset{n\in\mathbb{N}}{\cap} \overline{f_{[\alpha]_n}(B)} = Im\ a_\alpha$$

and

$$\lim_{n\to\infty} h(\overline{f_{[\alpha]_n}(B)}, Im\ a_\alpha) = 0.$$

b) If $\alpha \in \Lambda(I\cup J) \setminus \Sigma_1(I,J)$, *i.e.* $n_I(\alpha) < \infty$, *then*

$$a_\alpha(B) = f_{\alpha_1\alpha_2\ldots\alpha_{n^*}}(a_{\alpha_{n^*+1}\ldots\alpha_m\ldots}(B)),$$

and

$$\overline{a_{\alpha_{n^*+1}\ldots\alpha_m\ldots}(B)} = \underset{p\in\mathbb{N}}{\cap}\overline{f_{\alpha_{n^*+1}\ldots\alpha_{n^*+p}}(B)} = \lim_{p\to\infty}\overline{f_{\alpha_{n^*+1}\ldots\alpha_{n^*+p}}(B)},$$

where $n^* = \{\begin{matrix} \max\{n\in\mathbb{N} \mid \alpha_n\in I\}, \textit{ if } n_I(\alpha)\neq 0 \\ 0, \quad \textit{otherwise} \end{matrix}$, *with the convention that* $f_{\alpha_1\alpha_2\ldots\alpha_{n^*}} = Id_X$ *if* $n^*=0$.

Proof. a) We can prove, via the mathematical induction method, that:

-
$$f_{[\alpha]_{n+1}}(B) \subseteq f_{[\alpha]_n}(B),$$
for every $n\in\mathbb{N}$;

-
$$diam(f_{[\alpha]_n}(B)) \leq a^{n_I([\alpha]_n)} diam(B),$$
for every $n\in\mathbb{N}$.

Consequently $\underset{n\in\mathbb{N}}{\cap}\overline{f_{[\alpha]_n}(B)}$ consists of a single point.

Since $f_{[\alpha]_{n+p}}(x)\in f_{[\alpha]_n}(B)$ for every $x\in B$ and $n,p\in\mathbb{N}$, we infer that

$$a_\alpha(x) = \lim_{p\to\infty} f_{[\alpha]_{n+p}}(x) \in \overline{f_{[\alpha]_n}(B)},$$

for every $n\in\mathbb{N}$, so

$$a_\alpha(x) \in \underset{n\in\mathbb{N}}{\cap}\overline{f_{[\alpha]_n}(B)}.$$

As, taking into account Proposition 3.4, b), a_α is constant, we have

$$\underset{n\in\mathbb{N}}{\cap}\overline{f_{[\alpha]_n}(B)} = Im\ a_\alpha.$$

Since

$$\lim_{n\to\infty} h(\overline{f_{[\alpha]_n}(B)}, \underset{n\in\mathbb{N}}{\cap}\overline{f_{[\alpha]_n}(B)}) \overset{\text{Proposition 2.5}}{=} 0,$$

we conclude that

$$\lim_{n\to\infty} h(\overline{f_{[\alpha]_n}(B)}, Im\ a_\alpha) = 0.$$

b) We have

$$\begin{aligned} a_\alpha(B) &= \{a_{\alpha_1\alpha_2\ldots\alpha_{n^*}\alpha_{n^*+1}\ldots\alpha_m\ldots}(x) \mid x\in B\} \overset{\text{Proposition 3.5}}{=} \\ &= \{f_{\alpha_1\alpha_2\ldots\alpha_{n^*}}(a_{\alpha_{n^*+1}\ldots\alpha_m\ldots}(x)) \mid x\in B\} \\ &= f_{\alpha_1\alpha_2\ldots\alpha_{n^*}}(a_{\alpha_{n^*+1}\ldots\alpha_m\ldots}(B)). \end{aligned}$$

As $\alpha_k\in J$ for every $k\in\mathbb{N}$, $k\geq n^*+1$, via [16], we deduce that

$$\overline{a_{\alpha_{n^*+1}\ldots\alpha_m\ldots}(B)} = \underset{m\in\mathbb{N}}{\cap}\overline{f_{\alpha_{n^*+1}\ldots\alpha_{n^*+m}}(B)} = \lim_{m\to\infty}\overline{f_{\alpha_{n^*+1}\ldots\alpha_{n^*+m}}(B)}.$$

□

Lemma 3.10. *For every mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$A_{a_\alpha(x)} \subseteq A_x,$$

for all $x\in X$ *and* $\alpha\in\Lambda(I\cup J)$.

Proof. Since $\lim_{n\to\infty} f_{[\alpha]_n}(x) = a_\alpha(x)$, we obtain

$$A_{a_\alpha(x)} \overset{\text{Lemma 2.13}}{=} \lim_{n\to\infty} A_{f_{[\alpha]_n}(x)} \overset{\text{Proposition 2.4}}{=} \Big\{ y \in X \mid \text{there exist a strictly increasing sequence } (n_k)_{k\in\mathbb{N}} \text{ of natural numbers and } y_{n_k} \in A_{f_{[\alpha]_{n_k}}(x)} \overset{\text{Lemma 2.14}}{\subseteq} A_x \text{ for every } k \in \mathbb{N} \text{ such that } y = \lim_{k\to\infty} y_{n_k} \Big\} \subseteq \overline{A_x} = A_x,$$

for all $x \in X$ and $\alpha \in \Lambda(I \cup J)$. □

For an mIIFS $\mathcal{S} =((X, d),(f_i)_{i\in I\cup J})$, $B \in P_{b,cl}(X)$ and $\alpha \in \Sigma(I, J)$, we introduce the function $\mathcal{A}_\alpha : X \to X$ as follows:

-
$$\mathcal{A}_\alpha = a_\alpha,$$
if $\alpha \in \Sigma_1(I, J)$

-
$$\mathcal{A}_\alpha = f_{\beta_0} \circ a_{\gamma_1} \circ f_{\beta_1} \circ a_{\gamma_2} \circ ... \circ f_{\beta_{n-1}} \circ a_{\gamma_n} \circ f_{\beta_n},$$
if $\alpha = \beta_0\gamma_1\beta_1...\gamma_n\beta_n \in \Sigma_0(I, J)$, where $n \in \mathbb{N}$, $\beta_0 \in \{\lambda\} \cup \Lambda_1(I \cup J)$, $\beta_n \in \{\lambda\}\cup\Lambda_2(I\cup J)$, $\gamma_k \in \Lambda(J)$ for every $k \in \{1, ..., n\}$ and $\beta_k \in \Lambda_3(I\cup J)$ for every $k \in \{1, ..., n-1\}$ if $n \geq 2$.

Note that $\mathcal{A}_\alpha$ is well defined.

We also consider

$$\{\mathcal{A}_\alpha(x) \mid \alpha \in \Sigma(I, J) \text{ and } x \in B\} \overset{not}{=} L_B.$$

Note that, in view of the next Lemma, $L_B \in P_b(X)$.

Lemma 3.11. *For every mIIFS $\mathcal{S} =((X, d),(f_i)_{i\in I\cup J})$ and $B \in P_{b,cl}(X)$, we have*

$$\overline{L_B} \subseteq A_B.$$

Proof. One the one hand, we have

$$\mathcal{A}_\alpha(x) \overset{\text{Proposition 3.3}}{\in} A_B, \tag{1}$$

for all $x \in B$ and $\alpha \in \Sigma_1(I, J)$.

On the other hand, for all $x \in B$ and $\alpha = \beta_0\gamma_1\beta_1...\gamma_n\beta_n \in \Sigma_0(I, J)$, where $n \in \mathbb{N}$, $\beta_0 \in \{\lambda\} \cup \Lambda_1(I \cup J)$, $\beta_n \in \{\lambda\} \cup \Lambda_2(I \cup J)$, $\gamma_k \in \Lambda(J)$ for every $k \in \{1, ..., n\}$ and $\beta_k \in \Lambda_3(I \cup J)$ for every $k \in \{1, ..., n-1\}$ if $n \geq 2$, we have

$$a_{\gamma_n}(f_{\beta_n}(x)) \overset{\text{Proposition 3.3}}{\in} A_{f_{\beta_n}(x)} \overset{\text{Lemma 2.14}}{\subseteq} A_x,$$

so

$$a_{\gamma_{n-1}}(f_{\beta_{n-1}}(a_{\gamma_n}(f_{\beta_n}(x)))) \overset{\text{Proposition 3.3}}{\in} A_{f_{\beta_{n-1}}(a_{\gamma_n}(f_{\beta_n}(x)))} \subseteq \overset{\text{Lemma 2.14}}{\subseteq} A_{a_{\gamma_n}(f_{\beta_n}(x))} \overset{\text{Lemma 3.10}}{\subseteq} A_{f_{\beta_n}(x)} \overset{\text{Lemma 2.14}}{\subseteq} A_x,$$

and, inductively, we obtain

$$(f_{\beta_0} \circ a_{\gamma_1} \circ f_{\beta_1} \circ a_{\gamma_2} \circ ... \circ f_{\beta_{n-1}} \circ a_{\gamma_n} \circ f_{\beta_n})(x) \overset{\text{Proposition 3.5}}{=}$$
$$= (a_{\beta_0\gamma_1} \circ f_{\beta_1} \circ a_{\gamma_2} \circ ... \circ f_{\beta_{n-1}} \circ a_{\gamma_n} \circ f_{\beta_n})(x) \in A_x.$$

Hence

$$\mathcal{A}_\alpha(x) \in A_x \overset{\text{Proposition 3.8}}{\subseteq} A_B, \tag{2}$$

for all $x \in B$ and $\alpha \in \Sigma_0(I, J)$.

Therefore, via (1) and (2), we get $L_B \subseteq A_B$, and consequently $\overline{L_B} \subseteq A_B$. □

The next result provides an alternative description of the fixed points of the fractal operator associated with an mIIFS.

Theorem 3.12. *For every mIIFS* $\mathcal{S}$ =$((X, d), (f_i)_{i\in I\cup J})$ *and* $B \in P_{b,cl}(X)$*, we have*

$$\overline{L_B} = A_B,$$

i.e.

$$\overline{\{\mathcal{A}_\alpha(x) \mid \alpha \in \Sigma(I, J) \text{ and } x \in B\}} = A_B.$$

Proof. Let $B \in P_{b,cl}(X)$ be arbitrarily chosen, but fixed.

We will prove that

$$D(A_B, \overline{L_B}) = 0. \tag{*}$$

In view of Remark 3.7 and of the following inequality

$$D(A_B, \overline{L_B}) \overset{\text{Proposition 3.8}}{=} D(\overline{\underset{x\in B}{\cup} A_x}, \overline{L_B}) \overset{\text{Remark 2.1, i)}}{=}$$
$$= D(\underset{x\in B}{\cup} A_x, \overline{L_B}) \overset{\text{Remark 2.1, ii)}}{\leq} \sup_{x\in B} D(A_x, \overline{L_B})$$
$$\leq \sup_{x\in B} D(A_x, F_{\mathcal{S}}^{[n]}(\{x\})) + \sup_{x\in B} D(F_{\mathcal{S}}^{[n]}(\{x\}), \overline{L_B})$$
$$\leq \sup_{x\in B} h(A_x, F_{\mathcal{S}}^{[n]}(\{x\})) + \sup_{x\in B} D(F_{\mathcal{S}}^{[n]}(\{x\}), \overline{L_B}),$$

which is valid for all $n \in \mathbb{N}$, it suffices to prove that

$$\lim_{n\to\infty} \sup_{x\in B} D(F_{\mathcal{S}}^{[n]}(\{x\}), \overline{L_B}) = 0. \tag{1}$$

In order to justify (1), let us consider $\varepsilon > 0$ fixed, but arbitrarily chosen.
Then there exists $n_1 \in \mathbb{N}$ such that

$$\frac{a^{[\sqrt{n}]-2}}{1-a} N_B < \frac{\varepsilon}{2}, \tag{2}$$

for every $n \in \mathbb{N}$, $n \geq n_1$, where $a \in [0, 1)$ is the constant associated with $\mathcal{S}$ via Definition 2.11.

Now let us consider $n \in \mathbb{N}$, $n \geq n_1$, $x \in B$ and $z \in F_{\mathcal{S}}^{[n]}(\{x\})$.

Taking into account Proposition 2.8, there exists $\alpha = \alpha_1 ... \alpha_n \in \Lambda^*(I \cup J)$ such that

$$d(z, f_\alpha(x)) < \frac{\varepsilon}{2}. \tag{3}$$

The following two cases occur:

i) $n_I(\alpha) \geq [\sqrt{n}]$
ii) $n_I(\alpha) < [\sqrt{n}]$.

In the first case, we consider $\gamma \in \Lambda(I)$ and we have

$$d(f_{[\alpha\gamma]_{|\alpha|+k}}(x), f_{[\alpha\gamma]_{|\alpha|+k+1}}(x)) \overset{\text{Definition 2.11, b 1)}}{\leq} a^{n_I(\alpha)+k} N_B, \tag{4}$$

for every $k \in \mathbb{N}$, so

$$\begin{aligned} d(f_\alpha(x), \mathcal{A}_{\alpha\gamma}(x)) &\leq d(f_\alpha(x), f_{[\alpha\gamma]_{|\alpha|+l+1}}(x)) + d(f_{[\alpha\gamma]_{|\alpha|+l+1}}(x), \mathcal{A}_{\alpha\gamma}(x)) \\ &\leq \sum_{k=0}^{l} d(f_{[\alpha\gamma]_{|\alpha|+k}}(x), f_{[\alpha\gamma]_{|\alpha|+k+1}}(x)) + d(f_{[\alpha\gamma]_{|\alpha|+l+1}}(x), \mathcal{A}_{\alpha\gamma}(x)) \\ &\overset{(4)}{\leq} a^{n_I(\alpha)} N_B \sum_{k=0}^{l} a^k + d(f_{[\alpha\gamma]_{|\alpha|+l+1}}(x), \mathcal{A}_{\alpha\gamma}(x)) \\ &\leq \frac{a^{n_I(\alpha)}}{1-a} N_B + d(f_{[\alpha\gamma]_{|\alpha|+l+1}}(x), \mathcal{A}_{\alpha\gamma}(x)), \end{aligned}$$

for every $l \in \mathbb{N}$. By passing to the limit, as l goes to ∞, in the previous inequality, we get

$$d(f_\alpha(x), \mathcal{A}_{\alpha\gamma}(x)) \leq \frac{a^{n_I(\alpha)}}{1-a} N_B \leq \frac{a^{[\sqrt{n}]-2}}{1-a} N_B \overset{(2)}{<} \frac{\varepsilon}{2}. \tag{5}$$

Hence we have

$$\begin{aligned} d(z, \overline{L_B}) &\overset{\mathcal{A}_{\alpha\gamma}(x) \in L_B}{\leq} d(z, \mathcal{A}_{\alpha\gamma}(x)) \\ &\leq d(z, f_\alpha(x)) + d(f_\alpha(x), \mathcal{A}_{\alpha\gamma}(x)) \overset{(3)\ \&\ (5)}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{6}$$

In the second case, there exist $k \in \mathbb{N}$, $\beta_1, ..., \beta_{k+1} \in \Lambda^*(I)$ and $\gamma_j \in \Lambda^*(J) \setminus \{\lambda\}$, $j \in \{1, ..., k\}$ such that

$$\alpha = \beta_1 \gamma_1 \beta_2 \gamma_2 ... \beta_k \gamma_k \beta_{k+1}.$$

On the one hand, $|\beta_1| + |\beta_2| + ... + |\beta_{k+1}| = n_I(\alpha) < [\sqrt{n}]$ and $|\beta_1| + |\beta_2| + ... + |\beta_{k+1}| + |\gamma_1| + |\gamma_2| + ... + |\gamma_k| = n$, so

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_k| > n - [\sqrt{n}]. \tag{7}$$

On the other hand, as $k - 1 \leq n_I(\alpha) < [\sqrt{n}]$, we have

$$k \leq [\sqrt{n}] + 1. \tag{8}$$

Based on (7) and (8), via the generalized pigeonhole principle, there exists $j \in \{1, ..., k\}$ such that

$$|\gamma_j| \geq [\frac{|\gamma_1| + |\gamma_2| + ... + |\gamma_k| - 1}{k}] + 1 \geq \frac{|\gamma_1| + |\gamma_2| + ... + |\gamma_k| - 1}{k} \overset{(7)\ \&\ (8)}{\geq}$$
$$\geq \frac{n - [\sqrt{n}] - 1}{[\sqrt{n}] + 1} \geq \frac{([\sqrt{n}])^2 - [\sqrt{n}] - 1}{[\sqrt{n}] + 1} = [\sqrt{n}] - 2 + \frac{1}{[\sqrt{n}] + 1} \geq [\sqrt{n}] - 2. \tag{9}$$

In the sequel, we shall use the following notation:

$$(f_{\beta_{j+1}} \circ f_{\gamma_{j+1}} \circ ... \circ f_{\beta_k} \circ f_{\gamma_k} \circ f_{\beta_{k+1}})(x) \overset{not}{=} y.$$

For a fixed $\gamma = \gamma^1\gamma^2...\gamma^n... \in \Lambda(J)$, let us consider

$$\theta = \beta_1\gamma_1\beta_2\gamma_2...\beta_j\gamma_j\gamma\beta_{j+1}\gamma_{j+1}...\beta_k\gamma_k\beta_{k+1} \in \Sigma_0(I, J)$$

and

$$\theta_l = \beta_1\gamma_1\beta_2\gamma_2...\beta_j\gamma_j[\gamma]_l\beta_{j+1}\gamma_{j+1}...\beta_k\gamma_k\beta_{k+1},$$

where $l \in \{0\} \cup \mathbb{N}$. Then

$$\begin{aligned} d(f_\alpha(x), \mathcal{A}_\theta(x)) &\leq d(f_{\theta_0}(x), f_{\theta_l}(x)) + d(f_{\theta_l}(x), \mathcal{A}_\theta(x)) \\ &\leq \sum_{s=0}^{l-1} d(f_{\theta_s}(x), f_{\theta_{s+1}}(x)) + d(f_{\theta_l}(x), \mathcal{A}_\theta(x)) \\ &\overset{(9)}{\leq} \sum_{s=0}^{l-1} a^{[\sqrt{n}]-2+s} d(y, f_{\gamma^{s+1}}(y)) + d(f_{\theta_l}(x), \mathcal{A}_\theta(x)) \\ &\leq \sum_{s=0}^{l-1} a^{[\sqrt{n}]-2+s} N_B + d((f_{\beta_1} \circ f_{\gamma_1} \circ ... \circ f_{\gamma_j} \circ f_{[\gamma]_l})(y), \\ &\quad (f_{\beta_1} \circ f_{\gamma_1} \circ ... \circ f_{\gamma_j} \circ a_\gamma)(y)) \\ &\leq N_B \sum_{s=0}^{l-1} a^{[\sqrt{n}]-2+s} + d(f_{[\gamma]_l}(y), a_\gamma(y)) \leq \frac{a^{[\sqrt{n}]-2}}{1-a} N_B + d(f_{[\gamma]_l}(y), a_\gamma(y)), \end{aligned} \tag{10}$$

for every $l \in \{0\} \cup \mathbb{N}$, so, by passing to limit, as l goes to ∞, in (10), we infer that

$$d(f_\alpha(x), \mathcal{A}_\theta(x)) \leq \frac{a^{[\sqrt{n}]-2}}{1-a} N_B \overset{(2)}{<} \frac{\varepsilon}{2}. \tag{11}$$

As in the first case, we have

$$\begin{aligned} d(z, \overline{L_B}) &\overset{\mathcal{A}_\theta(x) \in L_B}{\leq} d(z, \mathcal{A}_\theta(x)) \\ &\leq d(z, f_\alpha(x)) + d(f_\alpha(x), \mathcal{A}_\theta(x)) \overset{(3)\ \&\ (11)}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{12}$$

Via (6) and (12), we conclude that

$$D(F_S^{[n]}(x), \overline{L_B}) = \sup_{z \in F_S^{[n]}(x)} d(z, \overline{L_B}) \leq \varepsilon,$$

for every $n \in \mathbb{N}$, $n \geq n_1$ and every $x \in B$, i.e.

$$\lim_{n\to\infty} \sup_{x\in B} D(F_S^{[n]}(x), \overline{L_B}) = 0,$$

so $(*)$ is proved and therefore

$$A_B \subseteq \overline{L_B}.$$

The above inclusion, together with Lemma 3.11, completes the proof. □

4. The Canonical Projection Associated with an mIIFS

Finally let us introduce the concept of canonical projection associated with an mIIFS and rewrite part of the obtained results.

Definition 4.1. *For an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, the function $\pi : \Sigma(I,J)\times X \to X$, given by*

$$\pi(\alpha, x) = \mathcal{A}_\alpha(x),$$

for every $(\alpha, x) \in \Sigma(I,J) \times X$, is called the canonical projection associated with $\mathcal{S}$.

Proposition 3.4, b) takes the following form (which is a counterpart of Theorem 2.9, b)):

Proposition 4.2. *For each mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, the set $\pi(\alpha, X)$ has just one element for all $\alpha \in \Sigma_1(I,J)$.*

The following result is the counterpart of Theorem 2.9, c), i).

Proposition 4.3. *For each mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, we have*

$$\pi \circ \tau_i = f_i \circ \pi,$$

for every $i \in I \cup J$, where $\tau_i : \Sigma(I,J)\times X \to \Sigma(I,J)\times X$ is defined by

$$\tau_i(\alpha, x) = (i\alpha, x),$$

for every $(\alpha, x) \in \Sigma(I,J) \times X$, i.e. the following diagram is commutative

$$\begin{array}{ccc} \Sigma(I,J)\times X & \overset{\tau_i}{\to} & \Sigma(I,J)\times X \\ \pi \downarrow & & \downarrow \pi \\ X & \overset{f_i}{\to} & X \end{array} .$$

Proof. For $\alpha \in \Sigma_1(I,J)$ the conclusion results from Proposition 3.5.

For $\alpha \in \Sigma_0(I,J)$, as $i\alpha \in \Sigma_0(I,J)$, the conclusion takes the following obvious form:

$$(f_{i\beta_0} \circ a_{\gamma_1} \circ f_{\beta_1} \circ a_{\gamma_2} \circ ... \circ f_{\beta_{n-1}} \circ a_{\gamma_n} \circ f_{\beta_n})(x)$$
$$= f_i((f_{\beta_0} \circ a_{\gamma_1} \circ f_{\beta_1} \circ a_{\gamma_2} \circ ... \circ f_{\beta_{n-1}} \circ a_{\gamma_n} \circ f_{\beta_n})(x)),$$

where $\alpha = \beta_0\gamma_1\beta_1...\gamma_n\beta_n \in \Sigma_0(I,J)$, with $n \in \mathbb{N}$, $\beta_0 \in \{\lambda\} \cup \Lambda_1(I \cup J)$, $\beta_n \in \{\lambda\} \cup \Lambda_2(I \cup J)$, $\gamma_k \in \Lambda(J)$ for every $k \in \{1,...,n\}$ and $\beta_k \in \Lambda_3(I \cup J)$ for every $k \in \{1,...,n-1\}$ if $n \geq 2$. □

Theorem 3.12 can be rewritten in the following form, which is the counterpart of Theorem 2.9, c), ii):

Theorem 4.4. *Let* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ *be a mIIFS and* $B \in P_{b,cl}(X)$*. Then*

$$\overline{\pi(\Sigma(I,J) \times B)} = A_B.$$

5. Visual Aspects Concerning the Functions A_α

Let us consider the mIIFS $\mathcal{S} = ((\mathbb{R}^2, \|.\|_2),(f_i)_{i\in I\cup J})$, where $I = \{1,2,3\}$, $J = \{4\}$ and $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$f_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),$$
$$f_2(x,y) = \left(\frac{x+1}{2}, \frac{y}{2}\right),$$
$$f_3(x,y) = \left(\frac{2x+1}{4}, \frac{2y+\sqrt{3}}{4}\right)$$

and

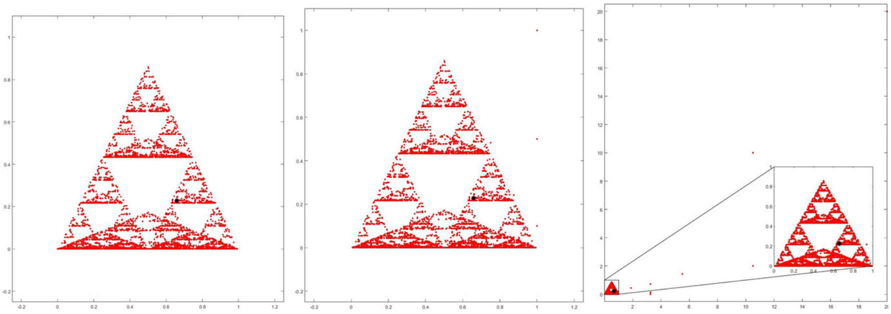
$$f_4(x,y) = \left(x, \frac{1}{5}y\right),$$

for every $(x,y) \in \mathbb{R}^2$.

A. We start by providing a visualization of the convergence of the sequence defining $a_\alpha(u)$ for a randomly generated $\alpha \in \Sigma_1(I,J)$ and for $u \in \{(0,0),(1,1),(20,20)\}$. More precisely, we represent the first 50,000 terms of the corresponding sequences, the limit being marked by * (see Figure 1).

Let us remark that the visualizations presented in Figure 1 are in accordance with Proposition 3.4, b), as all the three pictures indicate a convergence to the same limit.

B. Next we come up with a visualization of $\mathcal{A}_\alpha(u)$ for $u \in \{(0,0),(1,1)\}$ and for $\alpha \in \{\omega, 321\omega, \omega123, 12\omega31\} \subseteq \Sigma_1(I,J)$, where ω is the word having all letters equal to 4 (see Figure 2).



(a) $a_\alpha((0,0))$ (b) $a_\alpha((1,1))$ (c) $a_\alpha((20,20))$

FIGURE 1.

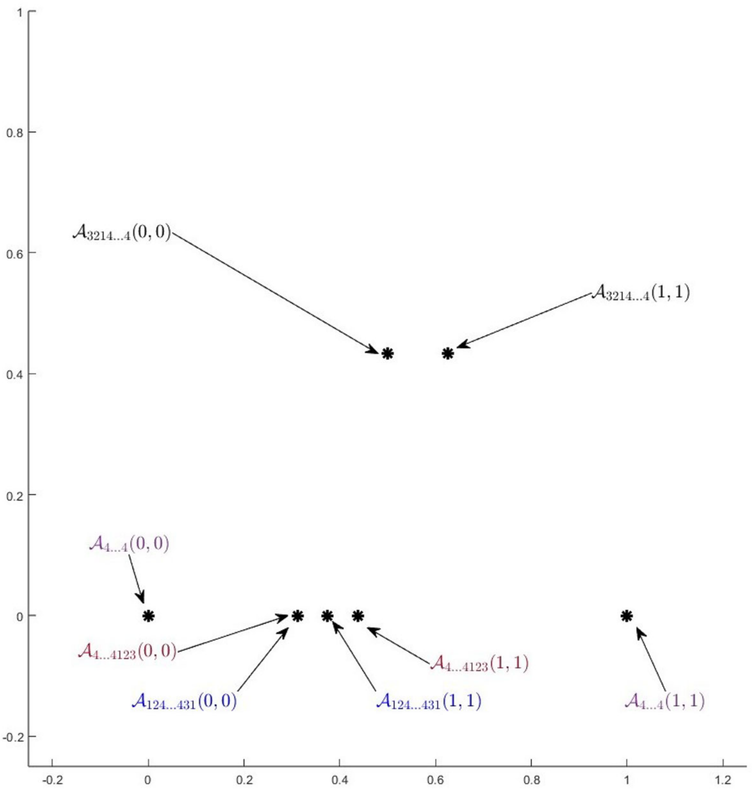
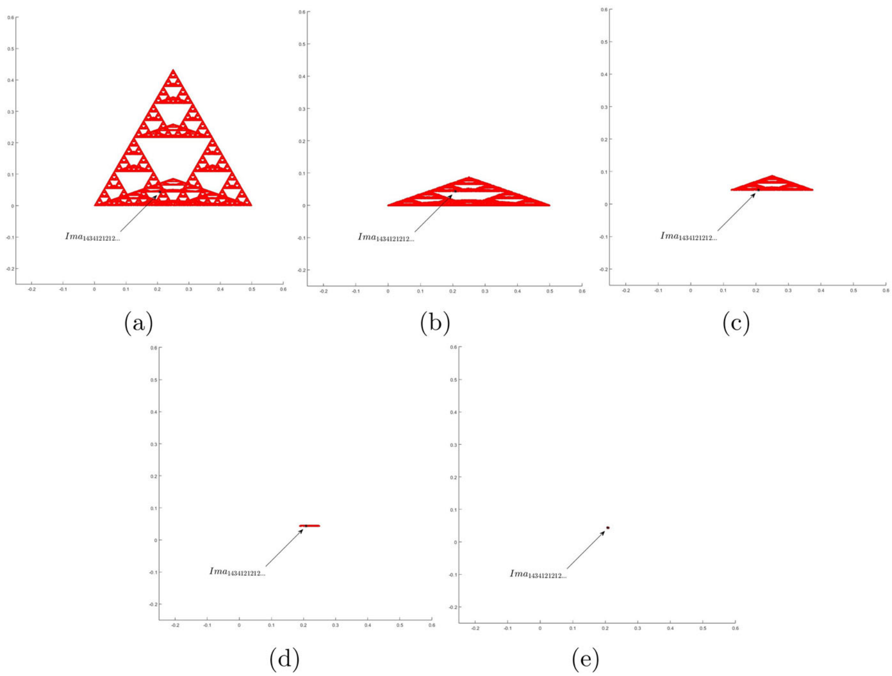


FIGURE 2.



(a) (b) (c) (d) (e)

FIGURE 3.

Let us remark that Figure 2 gives evidence that $\mathcal{A}_\alpha(u)$ depends on u.

C. Let us consider the set $B = F_{\mathcal{S}}^{[20]}(\{(0,0)\})$, $\alpha = 1434121212... \in \Sigma_1(I,J)$ and $\beta = 412\omega \in \Sigma_0(I,J)$.

Figures 3(a), (b), (c), (d) and (e) contain the visual representations of $f_{[\alpha]_n}(B)$ for $n=1$, $n=2$, $n=3$, $n=6$ and $n=9$, respectively. The fact that $f_{[\alpha]_9}(B)$ is very close with Ima_α (which is marked with *) endorses Theorem 3.9, a).

Figures 4(a), 4(b), 4(c), 4(d) and 4(e) contain the visual representations of $f_{[\beta]_n}(B)$ for $n=1$, $n=2$, $n=3$, $n=4$ and $n=6$, respectively and they illustrate Theorem 3.9, b) (the black horizontal segment representing $a_\beta(B)$).

D. Finally we note that the concatenation of the images of the approximates of $a_\alpha((0,0))$ for 50,000 randomly generated elements $\alpha \in \Sigma_1(I,J)$ with the images of $\mathcal{A}_\alpha((0,0))$ for another 50,000 randomly generated elements $\alpha \in \Sigma_0(I,J)$ (see Figure 5(a)) looks similar to the graphical representation of $F_{\mathcal{S}}^{[20]}(\{(0,0)\})$ (see Figure 5(b)). This remark is in accordance with Theorem 3.12.

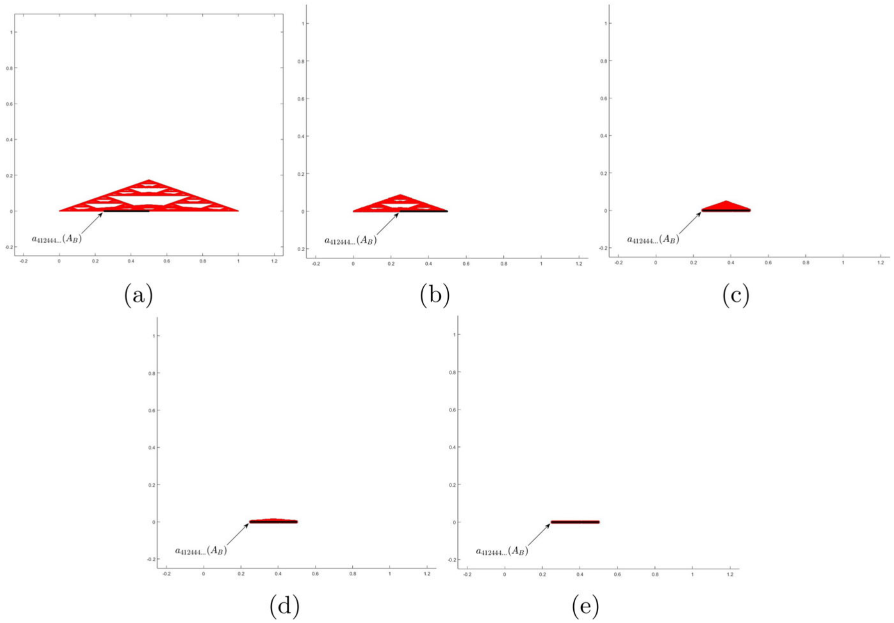


FIGURE 4.

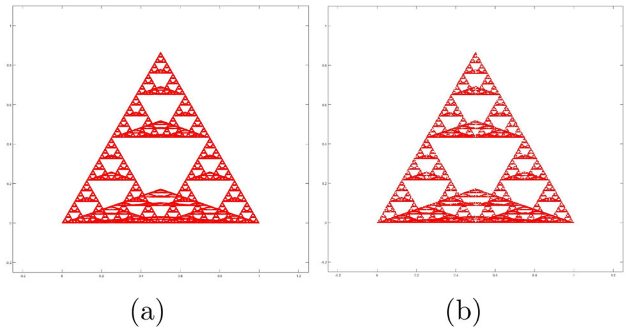


FIGURE 5.

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References

[1] Anghelina, B., Miculescu, R., Mihail, A.: On the fractal operator of a mixed possibly infinite iterated function system, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **119**, 31 (2025)

[2] Fernau, H.: Infinite iterated function systems. Math. Nachr. **170**, 79–91 (1994)

[3] Gwóźdź- Łukowska, G., Jachymski, J.: The Hutchinson-Barnsley theory for infinite iterated function systems. Bull. Aust. Math. Soc. **72**, 441–454 (2005)

[4] Hille, M.: Remarks on limits sets of infinite iterated functions systems. Monatsh. Math. **168**, 215–237 (2012)

[5] Hutchinson, J.: Fractals and self similarity. Indiana Univ. Math. J. **30**, 713–747 (1981)

[6] Iosifescu, M.: Iterated function systems. A critical survey. Math. Rep. (Bucur.) **11**, 181–229 (2009)

[7] Kameyama, A.: Distances on topological self-similar sets and the kneading determinants. J. Math. Kyoto Univ. **40**, 601–672 (2000)

[8] Kieninger, B.: Iterated Function Systems on Compact Hausdorff Spaces, Ph. D. thesis, University of Augsburg, Aachen: Shaker-Verlag, (2002)

[9] Kunze, H., La Torre, D., Mendivil, F., Vrscay, E.: Fractal-Based Methods in Analysis, Springer, (2012)

[10] Leśniak, K.: Infinite iterated function systems: a multivalued approach. Bull. Pol. Acad. Sci. Math. **52**, 1–8 (2004)

[11] Leśniak, K., Snigireva, N., Strobin, F.: Weakly contractive iterated function systems and beyond: a manual. J. Differ. Equ. Appl. **26**, 1114–1173 (2020)

[12] Mauldin, D., Urbański, M.: Dimensions and measures in infinite iterated function systems. Proc. Lond. Math. Soc. **73**, 105–154 (1996)

[13] Mauldin, D., Urbański, M.: Graph directed Markov Systems, Cambridge University Press, (2003)

[14] Mendivil, F.: A generalization of IFS with probabilities to infinitely many maps. Rocky Mountain J. Math. **28**, 1043–1051 (1998)

[15] Mihail, A., Miculescu, R.: The shift space for an infinite iterated function system. Math. Rep. Bucur. **61**, 21–32 (2009)

[16] Mihail, A., Savu, I.: φ-contractive parent-child possibly infinite IFSs and orbital φ-contractive possibly infinite IFSs. Fixed Point Theory **25**, 229–248 (2024)

[17] Strobin, F.: Contractive iterated function systems enriched with nonexpansive maps. Result. Math. **76**, 153 (2021)

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