Cent. Eur. J. Math. • 7(3) • 2009 • 529-538
DOI: 10.2478/s11533-009-0019-2

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Central European Journal of **Mathematics**

Two fixed point theorems for generalized contractions with constants in complete metric space

Research Article

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Received 3 November 2008; accepted 25 March 2009

Abstract: In this paper we prove two fixed point theorems for generalized contractions with constants in complete metric space, which are generalizations of very recent results of Kikkawa and Suzuki.

MSC: 54H25

Keywords: Fixed point • Contraction • Ćirić generalized contraction

1. Introduction

Let (X, d) be a complete metric space and T a selfmap of X. Then T is called a *contraction* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$.

T is called *Kannan* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$$

for all $x, y \in X$.

T is called *Chatterjea* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx)$$

for all $x, y \in X$.

We know that if X is complete, then every contraction, every Kannan mapping and every Chatterjea mapping have a unique fixed point, see [1, 2, 4].

Very recently, Suzuki [9] introduced a weaker notion of contraction and prove the following theorem, which is a new type of generalizations of the Banach contraction principle, and does characterize the metric completness, see also [10].

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Theorem 1.1 ([9]).
Define a nonincreasing function θ *from* $[0,1)$ *onto* $(1/2,1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)/r^2 & \text{if } (\sqrt{5}-1)/2 \le r \le 1/\sqrt{2}, \\ 1/(1+r) & \text{if } 1/\sqrt{2} \le r < 1. \end{cases} \tag{1}$$

Then for a metric space (X,d)*, the following are equivalent:*

(i) X *is complete.*

(ii) Every mapping T *on* X *satisfying the following has a fixed point:*
There exists $r \in [0,1)$ *such that* $\theta(r)d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le rd(x,y)$ *for all* $x,y \in X$.

Kikkawa and Suzuki gave the Kannan versions of Theorem 1.1. They proved the following theorems.

Theorem 1.2 ([5]).
Define a nonincreasing function θ *as in Theorem 1.1. Let* (X,d) *be a complete metric space and let* T *be a mapping on* X*. Suppose that there exists* $r \in [0,1)$ *such that* $\theta(r)d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le r\max\{d(x,Tx), d(y,Ty)\}$ *for all* $x,y \in X$*. Then* T *has a unique fixed point* z *and* $\lim_{n\to\infty} T^n x = z$ *holds for every* $x \in X$.

Theorem 1.3 ([5]).
Define a nonincreasing function θ *as in Theorem 1.1. Let* (X,d) *be a complete metric space. Let* S *and* T *be mappings on* X *satisfying the following:*

(a) S *is continuous;*

(b) $T(X) \subset S(X)$*;*

(c) S *and* T *commute.*

Suppose that there exists $r \in [0,1)$ *such that* $\theta(r)d(Sx,Tx) \le d(Sx,Sy)$ *implies* $d(Tx,Ty) \le rd(Sx,Sy)$ *for all* $x,y \in X$*. Then there exists a unique common fixed point of* S *and* T.

Popescu [8] gave a Chatterjea version of Theorem 1.1.

Theorem 1.4 ([8]).
Define a nonincreasing function θ *as in Theorem 1.1. Let* (X,d) *be a complete metric space and let* T *be a mapping on* X*. Let* $\alpha \in [0,1/2)$ *and put* $r := \alpha/(1-\alpha) \in [0,1)$*. Assume that*

$$\theta(r)d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le \alpha d(x,Ty) + \alpha d(y,Tx)$$

for all $x,y \in X$*. Then* T *has a unique fixed point* z *and* $\lim_{n\to\infty} T^n x = z$ *holds for every* $x \in X$.

Combining definitions of contraction, Kannan mapping and Chatterjea mapping, Zamfirescu [11] proved the following result.

Theorem 1.5 ([11]).
Let (X,d) *be a complete metric space and* $T : X \to X$ *a mapping for which there exists real numbers* a, b *and* c *satisfying* $a \in [0,1), b, c \in [0,1/2)$ *such that for each pair* $x,y \in X$*, at least one of the following condition hold:*

(z_1) $d(Tx,Ty) \le ad(x,y)$,

(z_2) $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$,

(z_3) $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n, n \in \mathbb{N}$, converges to p for any arbitrary but fixed $x_1 \in X$.

An operator T satisfying the contractive conditions $(z_1) - (z_3)$ is called Z-operator and can be written in the following equivalent form

$$d(Tx, Ty) \leq hm_T(x, y)$$

where $0 < h < 1$ and

$$m_T(x, y) = \max\{d(x, y), (d(x, Tx) + d(y, Ty))/2, (d(x, Ty) + d(y, Tx))/2\},$$

for all $x, y \in X$. Thus, the class of Z-operators is a subclass of mappings satisfying the following condition

$$d(Tx, Ty) \leq hM_T(x, y),$$

for all $x, y \in X$ where $0 < h < 1$ and

$$M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), (d(x, Ty) + d(y, Tx))/2\}.$$

The class of mappings satisfying the above condition was introduced and investigated by Ćirić [3] in 1971, and is commonly called a Ćirić generalized contraction. The purpose of this paper is to extend the results of Theorems 1.1 - 1.5 to the class of Ćirić generalized contractions and implicitely to the class of Z-operators.

2. Main result

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. For an arbitrary set A, we also denote by $\#A$ the number of elements of A. Now we can give a Ćirić version of Theorem 1.3.

Theorem 2.1.

Define a function θ as in Theorem 1.1. Let (X, d) be a complete metric space. Let S and T be mappings on X satisfying the following:

(a) S is continuous;

(b) $T(X) \subset S(X)$;

(c) S and T commute.

Suppose that there exists $r \in [0, 1)$ such that

$$\theta(r)d(Sx, Tx) \leq d(Sx, Sy) \text{ implies } d(Tx, Ty) \leq rM_{S,T}(x, y)$$

for all $x, y \in X$, where

$$M_{S,T}(x, y) = \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), (d(Sx, Ty) + d(Sy, Tx))/2\}. \tag{2}$$

Then there exists a unique common fixed point of S and T.

Proof. By (b), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$. Since $\theta(r) \leq 1$, $\theta(r)d(Sx, Tx) = \theta(r)d(Sx, SIx) \leq d(Sx, SIx)$ holds. From the assumption we have

$$\begin{aligned} d(SIx, SIIx) &= d(Tx, TIx) \\ &\leq rM_{S,T}(x, Ix) \\ &\leq r \max\{d(Sx, SIx), d(SIx, SIIx), d(Sx, SIIx)/2\} \\ &\leq r \max\{d(Sx, SIx), d(SIx, SIIx), (d(Sx, SIx) + d(SIx, SIIx))/2\} \\ &= r \max\{d(Sx, SIx), d(SIx, SIIx)\}. \end{aligned} \tag{3}$$

Therefore

$$d(SIx, SIIx) \leq rd(Sx, SIx) \tag{4}$$

holds for all $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = I^n u$ for all $n \in \mathbb{N}$. Then we note that $u_{n+1} = Iu_n$ and $Su_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. By (4) we have

$$\begin{aligned} d(Su_n, Su_{n+1}) = d(SIu_{n-1}, SIIu_{n-1}) &\leq rd(Su_{n-1}, SIu_{n-1}) \\ = rd(Su_{n-1}, Su_n) &\leq \ldots \leq r^n d(Su_0, Su_1) \end{aligned} \tag{5}$$

and hence $\sum_{n=1}^{\infty} d(Su_n, Su_{n+1}) < \infty$. Therefore $\{Su_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists a point $z \in X$ such that $Su_n \to z$. We also have $Tu_n = Su_{n+1} \to z$. We next show that

$$d(Tx, z) \leq r \max\{d(z, Sx), d(Sx, Tx)\} \tag{6}$$

for all $x \in X$ with $Sx \neq z$. Since $Su_n \to z$, $Tu_n \to z$ and $Sx \neq z$, there exists $\nu_1 \in \mathbb{N}$ such that $\theta(r)d(Su_n, Tu_n) \leq d(Su_n, Sx)$ for $n \geq \nu_1$ and hence by the assumption

$$d(Tu_n, Tx) \leq rM_{S,T}(u_n, x)$$

for $n \in \mathbb{N}$ with $n \geq \nu_1$. Therefore we get

$$\begin{aligned} d(Tx, z) &= \lim_n d(Tu_n, Tx) \leq \lim_n rM_{S,T}(u_n, x) \\ &= r \max\{d(z, Sx), d(z, z), d(Sx, Tx), (d(z, Tx) + d(z, Sx))/2\}. \end{aligned} \tag{7}$$

Then (6) holds for all $x \in X$ with $Sx \neq z$.
Let us prove that z is a fixed point of S. In the case where $\#\{n : d(Su_n, Tu_n) > d(Su_n, SSu_n)\} = \infty$, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $d(Su_{n_j}, Tu_{n_j}) > d(Su_{n_j}, SSu_{n_j})$. Then we have

$$\begin{aligned} d(Sz, z) &= \lim_{j\to\infty} d(SSu_{n_j}, z) \\ &\leq \lim_{j\to\infty} [d(SSu_{n_j}, Su_{n_j}) + d(Su_{n_j}, z)] \\ &\leq \lim_{j\to\infty} [d(Su_{n_j}, Tu_{n_j}) + d(Su_{n_j}, z)] \\ &= 0. \end{aligned} \tag{8}$$

In other case, where $\#\{n : d(Su_n, Tu_n) > d(Su_n, SSu_n)\} < \infty$, there exists $\nu_2 \in \mathbb{N}$ such that $d(Su_n, Tu_n) \leq d(Su_n, SSu_n)$ for all $n \geq \nu_2$. So, by the assumption we have

$$d(Tu_n, TSu_n) \leq rM_{S,T}(u_n, Su_n)$$

for all $n \geq \nu_2$. Since $TSu_n = STu_n \to Sz$ we obtain

$$\begin{aligned} d(z,Sz) &= \lim_n d(Tu_n, STu_n) = \lim_n d(Tu_n, TSu_n) \\ &\leq \lim_n rM_{S,T}(u_n, Su_n) \\ &= r\max\{d(z,Sz), d(z,z), d(Sz,Sz), (d(z,Sz)+d(Sz,z))/2\} \\ &= rd(z,Sz), \end{aligned} \tag{9}$$

and hence $d(z,Sz) = 0$. This implies that $z = Sz$. Therefore z is a fixed point of S in both cases.
We next show that

$$d(T^n z, T^{n+1}z) \leq rd(T^{n-1}z, T^n z), \tag{10}$$

for $n \in \mathbb{N}$ where $T^0 z = z$. Since

$$\begin{aligned} \theta(r)d(ST^{n-1}z, T^n z) \leq d(ST^{n-1}z, T^n z) &= d(ST^{n-1}z, T^n Sz) \\ &= d(ST^{n-1}z, ST^n z), \end{aligned} \tag{11}$$

we get

$$\begin{aligned} d(T^n z, T^{n+1}z) &\leq rM_{S,T}(T^{n-1}z, T^n z) \\ &= r\max\left\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z), d(T^{n-1}z, T^{n+1}z)/2\right\} \\ &\leq r\max\left\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z), (d(T^{n-1}z, T^n z) + d(T^n z, T^{n+1}z))/2\right\} \\ &= r\max\left\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\right\}. \end{aligned} \tag{12}$$

Thus (10) holds for every $n \in \mathbb{N}$. Using (10) we immediately can deduce by induction

$$d(T^n z, T^{n+1}z) \leq r^n d(Tz, z), \tag{13}$$

for all $n \in \mathbb{N}$.
We next show by induction that

$$d(T^n z, z) \leq d(Tz, z) \tag{14}$$

for all $n \in \mathbb{N}$. For $n = 1$ (14) is obvious. We suppose that $d(T^n z, z) \leq d(Tz, z)$ for some $n \in \mathbb{N}$. If $T^n z = z$ then $T^{n+1}z = Tz$ and $d(T^{n+1}z, z) = d(Tz, z) \leq d(Tz, z)$. If $T^n z \neq z$, since $ST^n z = T^n Sz = T^n z \neq z$, we have by (6) that

$$d(TT^n z, z) \leq r\max\{d(z, ST^n z), d(ST^n z, TT^n z)\}.$$

Thus

$$d(T^{n+1}z, z) \leq r\max\left\{d(z, T^n z), d(T^n z, T^{n+1}z)\right\},$$

and by (13) we get

$$\begin{aligned} d(T^{n+1}z, z) &\leq r\max\{d(z, T^n z), r^n d(Tz, z)\} \\ &\leq r\max\{d(Tz, z), r^n d(Tz, z)\} \\ &= rd(Tz, z) \leq d(Tz, z). \end{aligned} \tag{15}$$

By induction (14) holds for $n \in \mathbb{N}$.
Let us prove that z is a fixed point of T. We have two cases.

Case 1. In the case where $0 \le r < 1/\sqrt{2}$, we note $\theta(r) \le (1-r)/r^2$. We first prove that

$$d(T^n z, Tz) \le r d(Tz, z) \tag{16}$$

for all $n \in \mathbb{N}$. For $n = 1$ (16) is obvious. For $n = 2$ (16) becomes (13), so (16) holds. We suppose that $d(T^n z, Tz) \le r d(Tz, z)$ for some $n \in \mathbb{N}$ with $n \ge 2$. Since $d(z, Tz) \le d(z, T^n z) + d(T^n z, Tz) \le d(z, T^n z) + r d(z, Tz)$ we have $d(z, Tz) \le (1/(1-r)) d(z, T^n z)$ and hence

$$\begin{aligned} \theta(r) d(ST^n z, TT^n z) = \theta(r) d(T^n z, T^{n+1} z) &\le ((1-r)/r^2) d(T^n z, T^{n+1} z) \\ &\le ((1-r)/r^n) d(T^n z, T^{n+1} z) \le (1-r) d(Tz, z) \\ &\le d(z, T^n z) = d(Sz, ST^n z). \end{aligned} \tag{17}$$

Therefore by the assumption, we get

$$\begin{aligned} d(T^{n+1} z, Tz) &\le r M_{S,T}(z, T^n z) \\ &= r \max\left\{d(z, T^n z), d(z, Tz), d(T^n z, T^{n+1} z), (d(z, T^{n+1} z) + d(Tz, T^n z))/2\right\} \\ &= r d(z, Tz). \end{aligned} \tag{18}$$

By induction, (16) holds for $n \in \mathbb{N}$.
Arguing by contradiction, we assume $Tz \ne z$. Then by (16) $T^n z \ne z$ holds for all $n \in \mathbb{N}$, so $ST^n z = T^n Sz = T^n z \ne z$. Thus, by (6) we get

$$\begin{aligned} d(T^{n+1} z, z) &\le r \max\left\{d(ST^n z, z), d(ST^n z, T^{n+1} z)\right\} \\ &= r \max\left\{d(T^n z, z), d(T^n z, T^{n+1} z)\right\} \\ &\le r \max\left\{d(T^n z, z), r^n d(z, Tz)\right\}. \end{aligned} \tag{19}$$

Since $d(T^n z, z) \ge d(Tz, z) - d(T^n z, Tz) \ge d(Tz, z) - r d(Tz, z)$, we have $d(T^n z, z) \ge (1-r) d(Tz, z)$, so there exists $\nu_3 \in \mathbb{N}$ such that $d(T^n z, z) \ge r^n d(Tz, z)$ for $n \ge \nu_3$. Then by (19) for $n \ge \nu_3$

$$d(T^{n+1} z, z) \le r d(T^n z, z) \le \ldots \le r^{n-\nu_3+1} d(T^{\nu_3} z, z). \tag{20}$$

This implies $T^n z \to z$, which contradicts (16). Therefore we obtain $Tz = z$.

Case 2. In the case where $1/\sqrt{2} \le r < 1$, we can show that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\theta(r) d(Su_{n_j}, Su_{n_j+1}) \le d(Su_{n_j}, z)$ for $j \in \mathbb{N}$. Indeed, by (5) we have

$$d(Su_n, Su_{n+1}) = d(SIu_{n-1}, SIIu_{n-1}) \le r d(Su_{n-1}, SIu_{n-1}) = r d(Su_{n-1}, Su_n).$$

We assume $(1/(1+r)) d(Su_{n-1}, Su_n) > d(Su_{n-1}, z)$ and $(1/(1+r)) d(Su_n, Su_{n+1}) > d(Su_n, z)$. Then we have

$$\begin{aligned} d(Su_{n-1}, Su_n) &\le d(Su_{n-1}, z) + d(Su_n, z) \\ &< (1/(1+r))[d(Su_{n-1}, Su_n) + d(Su_n, Su_{n+1})] \\ &\le (1/(1+r))[d(Su_{n-1}, Su_n) + r d(Su_{n-1}, Su_n)] \\ &= d(Su_{n-1}, Su_n). \end{aligned} \tag{21}$$

This is a contradiction. Therefore, either $(1/(1+r)) d(Su_{n-1}, Su_n) \le d(Su_{n-1}, z)$ or $(1/(1+r)) d(Su_n, Su_{n+1}) \le d(Su_n, z)$. This implies that, either $\theta(r) d(Su_{2n-1}, Su_{2n}) \le d(Su_{2n-1}, z)$ or $\theta(r) d(Su_{2n}, Su_{2n+1}) \le d(Su_{2n}, z)$ holds for $n \in \mathbb{N}$. Thus, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\theta(r)d(Su_{n_j}, Su_{n_j+1}) \leq d(Su_{n_j}, z)$ for $j \in \mathbb{N}$. Then $\theta(r)d(Su_{n_j}, Tu_{n_j}) \leq d(Su_{n_j}, Sz)$ for $j \in \mathbb{N}$ and from the assumption, we have

$$\begin{aligned} d(z, Tz) &= \lim_{j\to\infty} d(Tu_{n_j}, Tz) \leq \lim_{j\to\infty} rM_{S,T}(u_{n_j}, z) \\ &= r\max\{d(z, Sz), d(z, z), d(Sz, Tz), (d(z, Tz) + d(Sz, z))/2\} \\ &= rd(z, Tz). \end{aligned} \tag{22}$$

Since $r < 1$, the above inequality implies that $Tz = z$. Therefore, we have shown that $Tz = z$ in both cases. From (6) we obtain the fixed point z is unique. □

Remark 2.1.

By [9] $\theta(r)$ is the best constant for every r.

Taking $S = I$ in Theorem 2.1, where I is identity on X, we obtain the following result.

Corollary 2.1.

Define a function θ as in Theorem 1.1. Let (X, d) be a complete metric space. Let T be mapping on X. Suppose that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rM_T(x, y)$$

for all $x, y \in X$, where

$$M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), (d(x, Ty) + d(y, Tx))/2\}.$$

Then there exists a unique fixed point of T.

3. The Meir-Keeler theorem

In this section we generalize another result of Kikkawa and Suzuki [5] and the Meir-Keeler theorem [7].

Theorem 3.1.

Let (X, d) be a complete metric space. Let S and T be mappings on X satisfying (a)-(c) in Theorem 2.1. Assume that

$$(1/2)d(Sx, Tx) < d(Sx, Sy) \text{ implies } d(Tx, Ty) < \max\{d(Sx, Sy), (d(Sx, Tx) + d(Sy, Ty))/2\} \tag{23}$$

for all $x, y \in X$, and that for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$(1/2)d(Sx, Tx) < d(Sx, Sy) \text{ and } \max\{d(Sx, Sy), (d(Sx, Tx) + d(Sy, Ty))/2\} < \epsilon + \delta(\epsilon) \text{ implies } d(Tx, Ty) \leq \epsilon. \tag{24}$$

for all $x, y \in X$. Then there exists a unique common fixed point of S and T.

Proof. By (b), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$, and $Ix = x$ for all $x \in X$ with $Sx = Tx$. For $x \in X$ with $Sx \neq Tx$, we have $d(Sx, Tx) < 2d(Sx, Tx) = 2d(Sx, SIx)$. It follows from (23) that

$$\begin{aligned} d(Tx, TIx) &< \max\{d(Sx, SIx), (d(Sx, Tx) + d(SIx, TIx))/2\} \\ &\leq \max\{d(Sx, Tx), d(Tx, TIx)\} \end{aligned} \tag{25}$$

and thus

$$d(SIx, SIIx) < d(Sx, SIx) \text{ for all } x \in X \text{ with } Sx \neq SIx. \tag{26}$$

For $x \in X$ with $Sx = Tx$, we have $Ix = x$ and then $d(SIx, SIIx) = d(Sx, SIx) = 0$. Therefore we obtain

$$d(SIx, SIIx) \leq d(Sx, SIx) \text{ for all } x \in X. \tag{27}$$

Let $u \in X$. Put $u_0 = u$ and $u_n = I^n u$ for all $n \in \mathbb{N}$. By (27), $\{d(Su_n, Su_{n+1})\}$ is a nonincreasing sequence and hence converges to some $\alpha \geq 0$. Suppose $\alpha > 0$. Then by (26), $\{d(Su_n, Su_{n+1})\}$ is strictly decreasing and hence $d(Su_n, Su_{n+1}) > \alpha$ for all $n \in \mathbb{N}$. Take $j \in \mathbb{N}$ with $d(Su_j, Su_{j+1}) < \alpha + \delta(\alpha)$. Since

$$\begin{aligned}
&\max\left\{d(Su_j, Su_{j+1}), (d(Su_j, Tu_j) + d(Su_{j+1}, Tu_{j+1}))/2\right\} \\
&= \max\left\{d(Su_j, Su_{j+1}), (d(Su_j, Su_{j+1}) + d(Su_{j+1}, Su_{j+2}))/2)\right\} \\
&= d(Su_j, Su_{j+1}),
\end{aligned} \tag{28}$$

it follows by (24) that $d(Su_{j+1}, Su_{j+2}) \leq \alpha$. This is a contradiction. Therefore we get $\alpha = 0$, so

$$\lim_{n\to\infty} d(Su_n, Su_{n+1}) = 0. \tag{29}$$

Fix $\epsilon > 0$ and put $\delta_1 = \min\{\epsilon, \delta(\epsilon)\}$. By (29), we can choose $\nu_1 \in \mathbb{N}$ such that $d(Su_n, Su_{n+1}) < \delta_1$ for all $n \geq \nu_1$. Fix $l \in \mathbb{N}$ with $l \geq \nu_1$. We shall show that

$$d(Su_l, Su_{l+m}) < \epsilon + \delta_1 \tag{30}$$

for all $m \in \mathbb{N}$ by induction. If $m = 1$, (30) is obvious. Suppose that $d(Su_l, Su_{l+m}) < \epsilon + \delta_1$ holds for some $m \in \mathbb{N}$. In the case where $d(Su_l, Su_{l+m}) \leq \epsilon$, we have

$$d(Su_l, Su_{l+m+1}) \leq d(Su_l, Su_{l+m}) + d(Su_{l+m}, Su_{l+m+1}) < \epsilon + \delta_1.$$

In the other case, where $\epsilon < d(Su_l, Su_{l+m}) < \epsilon + \delta_1$, we have $d(Su_l, Su_{l+1}) < \delta_1 \leq \epsilon < d(Su_l, Su_{l+m}) < 2d(Su_l, Su_{l+m})$. Moreover,

$$\begin{aligned}
&\max\{d(Su_l, Su_{l+m}), (d(Su_l, Su_{l+1}) + d(Su_{l+m}, Su_{l+m+1}))/2\} \\
&< \max\{\epsilon + \delta_1, (\delta_1 + \delta_1)/2\} = \epsilon + \delta_1 \\
&\leq \epsilon + \delta(\epsilon).
\end{aligned} \tag{31}$$

By (24) we obtain $d(Su_{l+1}, Su_{l+m+1}) \leq \epsilon$. Hence,

$$d(Su_l, Su_{l+m+1}) \leq d(Su_l, Su_{l+1}) + d(Su_{l+1}, Su_{l+m+1}) \leq \epsilon + \delta_1. \tag{32}$$

So, by induction, (30) holds for all $m \in \mathbb{N}$. Since ϵ is arbitrary, we have

$$\lim_{n\to\infty} \sup_{m>n} d(Su_m, Su_n) = 0.$$

Therefore $\{Su_n\}$ is a Cauchy sequence and by completness of X, there exists $z \in X$ such that $Su_n \to z$. Next we show that z is a fixed point of S. Arguing by contradiction, we assume that $Sz \neq z$. Then we denote by $\beta := d(Sz, z)$. Obviously, we have $\beta > 0$. Since $d(Su_n, Su_{n+1}) \to 0$, $d(SSu_n, SSu_{n+1}) \to 0$, $d(Su_n, SSu_n) \to \beta$, there exists $\nu_2 \in \mathbb{N}$

such that $d(Su_n, Su_{n+1}) < \beta/2$, $d(SSu_n, SSu_{n+1}) < \beta/2$, $d(Su_n, SSu_n) > \beta/2$ for all $n \geq \nu_2$. Then $d(Su_n, Tu_n)/2 = d(Su_n, Su_{n+1})/2 < \beta/4 < d(Su_n, SSu_n)$, so by (23) we obtain that

$$d(Tu_n, TSu_n) < \max\{d(Su_n, SSu_n), (d(Su_n, Tu_n) + d(SSu_n, TSu_n))/2\}$$

and hence

$$d(Su_{n+1}, SSu_{n+1}) < \max\{d(Su_n, SSu_n), (d(Su_n, Su_{n+1}) + d(SSu_n, SSu_{n+1}))/2\}\,. \tag{33}$$

Since $(d(Su_n, Su_{n+1}) + d(SSu_n, SSu_{n+1}))/2 < \beta/2 < d(Su_n, SSu_n)$, we get $d(Su_{n+1}, SSu_{n+1}) < d(Su_n, SSu_n)$, for all $n \geq \nu_2$. This implies that $\{d(Su_n, SSu_n)\}$ is strictly decreasing for large $n \in \mathbb{N}$ and $d(Su_n, SSu_n) > \beta$ for $n \geq \nu_2$. Then we can take $j \in \mathbb{N}$ such that $j \geq \nu_2$ and $d(Su_j, SSu_j) < \beta + \delta(\beta)$. Then $d(Su_j, Tu_j) = d(Su_j, Su_{j+1}) < \beta/2 < 2d(Su_j, SSu_j)$ and

$$d(Su_{j+1}, SSu_{j+1}) = \max\left\{d(Su_j, SSu_j), (d(Su_j, Tu_j) + d(SSu_j, TSu_j))/2\right\} < \beta + \delta(\beta). \tag{34}$$

From (24) and (c), we obtain $d(Su_{j+1}, SSu_{j+1}) = d(Tu_j, TSu_j) \leq \beta$. This is a contradiction. Thus, we obtain that $Sz = z$.

Let us prove $Tz = z$. If there exists $\nu \in \mathbb{N}$ such that $Su_\nu = Su_{\nu+1}$ then $Su_\nu = Tu_\nu$ and by definition of I we obtain $u_\nu = u_{\nu+1}$. Hence $u_n = u_\nu$ for all $n \geq \nu$. Since $Su_n \to z$, we have $Su_n = z$ for $n \geq \nu$ and then $Tz = TSu_\nu = STu_\nu = SSu_{\nu+1} = Sz = z$. In the other case, we have $Su_n \neq Su_{n+1}$ for all $n \in \mathbb{N}$, so $Su_n \neq Tu_n$ for $n \in \mathbb{N}$. If $d(Su_n, Su_{n+1}) \geq 2d(Su_n, z)$ and $d(Su_{n+1}, Su_{n+2}) \geq 2d(Su_{n+1}, z)$, then we have by (26)

$$\begin{aligned} d(Su_n, Su_{n+1}) &\leq d(Su_n, z) + d(Su_{n+1}, z) \\ &\leq (d(Su_n, Su_{n+1}) + d(Su_{n+1}, Su_{n+2}))/2 \\ &< d(Su_n, Su_{n+1}). \end{aligned} \tag{35}$$

This is a contradiction. Therefore we have either $d(Su_n, Su_{n+1}) < 2d(Su_n, z)$ or $d(Su_{n+1}, Su_{n+2}) < 2d(Su_{n+1}, z)$ for $n \in \mathbb{N}$. Then, from (23) either

$$d(Tu_n, Tz) < \max\{d(Su_n, Sz), (d(Su_n, Tu_n) + d(Sz, Tz))/2\} \tag{36}$$

or

$$d(Tu_{n+1}, Tz) < \max\{d(Su_{n+1}, Sz), (d(Su_{n+1}, Tu_{n+1}) + d(Sz, Tz))/2\}\,. \tag{37}$$

holds for $n \in \mathbb{N}$. Therefore, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$d(Tu_{n_j}, Tz) < \max\left\{d(Su_{n_j}, Sz), (d(Su_{n_j}, Tu_{n_j}) + d(Sz, Tz))/2\right\}\,. \tag{38}$$

holds for $j \in \mathbb{N}$. Since $Tu_{n_j} = Su_{n_j+1}$ and $Su_n \to z$, we obtain that

$$d(z, Tz) \leq \max\{d(z, Sz), (d(z, z) + d(Sz, Tz))/2\}\,. \tag{39}$$

But $Sz = z$, so $d(z, Tz) \leq d(z, Tz)/2$. This implies $Tz = z$. Hence, in all cases, we have shown z is a common fixed point of S and T.

Finally, we suppose that y is another common fixed point of S and T. Since $d(Sz, Tz)/2 = 0 < d(z, y) = d(Sx, Sy)$, we have by (23)

$$d(z, y) = d(Tz, Ty) < \max\{d(Sz, Sy), (d(Sz, Tz) + d(Sy, Ty))/2\} = d(Sz, Sy) = d(z, y). \tag{40}$$

This is a contradiction, so the common fixed point is unique. □

Taking $S = I$ in Theorem 3.1, where I is identity on X, we obtain

Corollary 3.1.

Let (X, d) be a complete metric space. Let T be mapping on X. Assume that

$$(1/2)d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) < \max\{d(x,y), (d(x,Tx)+d(y,Ty))/2\} \tag{41}$$

for all $x, y \in X$, and that for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$(1/2)d(x,Tx) < d(x,y) \text{ and } \max\{d(x,y), (d(x,Tx)+d(y,Ty))/2\} < \epsilon + \delta(\epsilon) \text{ implies } d(Tx,Ty) \le \epsilon. \tag{42}$$

for all $x, y \in X$. Then there exists a unique fixed point of T.

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